Problem 1. (20)
Let $G$ be a finite group and $p$ be a prime number such that $p$ divides the order of $G$. Let $n_p$ be the number of Sylow $p$-subgroups in $G$. If $n_p \not\equiv 1 \pmod{p^2}$, show that there are two Sylow $p$-subgroups $P$ and $Q$ such that $[P : P \cap Q] = p$. (Hint: Consider the action of $P$ on the set of all Sylow $p$-subgroups by conjugation.)
Problem 2. (30)
Let $R$ be a commutative ring with identity. Let $\mathcal{N}$ be the set of all nilpotent elements in $R$. Prove the following:

(a) $\mathcal{N}$ is an ideal of $R$.

(b) $\mathcal{N}$ is contained in the intersection of all prime ideals of $R$.

(c) $\mathcal{N}$ is exactly the intersection of all prime ideals of $R$. (Hint: If not, let $P_0$ be the intersection and $a \in P_0 - \mathcal{N}$. Let $\mathcal{M}$ be the partially ordered set consisting of ideals that do not contain any power of $a$. Show that $\mathcal{M}$ has a maximal element.)
Problem 3. (30)
Let $K/F$ be a finite Galois extension with Galois group of order $3393 = 3^2 \cdot 13 \cdot 29$. Show that there are intermediate subfields $E_1$, $E_2$, and $E_3$ such that

1. $E_0 = F \subset E_1 \subset E_2 \subset E_3 \subset E_4 = K$, where each containment is proper, and
2. $E_{i+1}/E_i$ is a Galois extension for $i = 0, 1, 2, 3$.

Determine the Galois groups $\{\text{Gal}(E_{i+1}/E_i) | i = 0, 1, 2, 3\}$ as a set.
Problem 4. (20)
Let $p$ be a prime number and $F$ be a finite field of order $p$.

(a) Show that if $a$ is a nonzero element in $F$, then the polynomial $f = x^p - x + a$ is irreducible over $F$. (Hint: If $\alpha$ is a zero of $f$, consider $\alpha + 1$.)

(b) Decide if the splitting fields of $f_1 = x^3 - x + 1$ and $f_2 = x^3 - x - 1$ over $F = \mathbb{Z}_3$ are isomorphic or not. If they are, find an explicit isomorphism between them. If not, explain why.