Math 53000 Qualifying Exam August 2014

Notation: unit disk: \( \mathbb{D} = \{ z : |z| < 1 \} \) and right half plane: \( H_+ = \{ z : \text{Re}(z) > 0 \} \).

1. (a) Give a careful statement of the Schwarz Lemma, including information about the derivative and equality in the inequalities, which begins: If \( \varphi : \mathbb{D} \rightarrow \mathbb{D} \) is an analytic function mapping the unit disk into itself such that \( \varphi(0) = 0 \), then \( \cdots \)

(b) Let \( f \) be an analytic function mapping the right half plane, \( H_+ \), into itself such that \( f(1) = 1 \).
   Prove that for \( w \) in \( H_+ \),
   \[
   \left| \frac{f(w) - 1}{f(w) + 1} \right| < \left| \frac{w - 1}{w + 1} \right|
   \]

(c) What can you say about \( f \) if there is \( w_0 \neq 1 \) for which equality holds in this inequality?

2. Find the number of roots of the polynomial \( p(z) = z^7 - 8z^2 + 2 \) in the annulus \( \{ z : 1 < |z| < 2 \} \).
   Prove that your answer is correct, citing appropriate theorems and justifying their hypotheses.

3. Let \( S = \{ z : \text{Re}(z) > 0 \text{ and } |\text{Im}(z)| < \frac{3\pi}{2} \} \) and let \( g(z) = e^z \) for \( z \) in \( S \).
   (a) Describe the set \( g(S) = \{ w \in \mathbb{C} : w = g(z) \text{ for some } z \text{ in } S \} \).
   (b) For any set \( F \), let \( \#(F) \) denote the number of elements of \( F \), so \( \#(\emptyset) = 0 \), \( \#(\mathbb{C}) = \infty \), and \( \#(\{n : n \text{ is an integer and } |n| \leq 10\}) = 21 \).
      For each point \( w \) in \( g(S) \), find \( \#(\{ z \in S : g(z) = w \}) \), that is, for each \( w \) in \( g(S) \), find the number of points of \( S \) that map to \( w \).

4. Let \( \Omega_1 = \{ z = x + iy : -3 < x < 2 \text{ and } y = 0 \} \cup \{ z = x + iy : x < 2 \text{ and } y \neq 0 \} \)
   and \( \Omega_2 = \{ z = x + iy : -3 < x < 2 \text{ and } y = 0 \} \cup \{ z = x + iy : x > -3 \text{ and } y \neq 0 \} \).
   The function \( \zeta \) is analytic on \( \Omega_1 \) and has a non-removable singularity at \(-3 \). Similarly, the function \( \eta \)
   is analytic on \( \Omega_2 \) and has a non-removable singularity at \( 2 \).
   Finally, these functions satisfy \( \zeta(-1) = \eta(-1) \) and the derivatives \( \zeta^{(k)}(-1) = \eta^{(k)}(-1) \) for all positive integers \( k \), where \( h^{(1)}(z) = h'(z) \), \( h^{(2)}(z) = h''(z) \), etc.
   (a) Find the coefficients \( a_k \) (in terms of information about \( \zeta \)) so that the power series for \( \zeta \) centered
   at \( c = -1 \) is \( \sum_{k=0}^{\infty} a_k (z + 1)^k \).
   (b) What is the radius of convergence of the power series for \( \zeta \) centered at \( a = -1 \)? Explain why!
   (c) Prove that \( \zeta(z) = \eta(z) \) for all \( z \) in \( \Omega_1 \cap \Omega_2 = \{ z = x + iy : -3 < x < 2 \} \).

5. Find the Laurent series for \( h(z) = \frac{z}{z^2 + 2z - 3} \) that converges on the open annulus \( \{ z : 1 < |z| < 3 \} \).

6. Use contour integration to compute \( \int_0^\infty \frac{x^{1/3}}{1 + x^2} \, dx \).
   Justify your calculation by explicitly describing the contour(s), the substitutions used in each piece
   of the contour(s), and the estimates necessary for drawing your conclusion. (You should express your
   answer in a form that is obviously a real number!)