IUPUI Department of Mathematical Sciences
Departmental Final Examination

PRACTICE FINAL EXAM VERSION #1

MATH 15400

Algebra & Trigonometry II

Exam directions similar to those on the departmental final.
1. **DO NOT OPEN** this test booklet until you are told to do so.
2. This is NOT the exam for MATH 15300 or 15900.
3. There are 7 pages in this exam with problems 1 to 23 and a bonus problem.
4. You MUST get a new exam from the proctor if your exam is incomplete.
5. PRINT your name and student ID# below.
6. MARK your section below.
7. You will have two hours to complete this examination.
8. A TI-30Xa calculator is permitted, no other calculator is allowed.
9. No scrap paper, notes, books, or collaborators are allowed.
10. Exact answers may contain π or radicals or logarithms.
11. Simplify all answers completely.
12. Problems involving units must have the units represented on the answer to receive full credit.

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Practice Departmental Final Exam Recommendations to Students:
- Take this practice final exam like an actual examination (not like doing homework). That is, create an “exam like” atmosphere. This practice exam should be taken after completing a thorough review of the material.
- Set aside a two-hour block of time with no interruptions (no facebook, texting, phone calls, restroom breaks, etc.).
- Do not use any help aids, such as notes, textbook, internet, scrap paper, MAC staff, etc.
- Work through all problems noting which concepts you know well and which ones you need to spend more time on.
- Grade your exam using the answers in the back of your textbook (the textbook section and exercise number is noted at the top right of each problem).
- Rework any problem on the exam that you missed and then work similar problems from the textbook until you can perform the operations without error.
- Follow the same recommendations for taking the Practice Final Exam Version #2.
1. Find the vertex, focus, and directrix of the parabola.

\[ y = x^2 - 4x + 2 \]

Vertical \((x-h)^2 = 4p(y-k)\)

\[
\begin{align*}
\text{Vertex: } & (h, k) \\
\text{Focus: } & (h, k+p) \\
\text{Directrix: } & y = k - p
\end{align*}
\]

\[ V(2, -2), F(2, -7/4) \]

1. \[ y = -9/4 \] (4)

2. Find an equation of the ellipse that has its center at the origin with horizontal major axis of length 8, minor axis of length 5.

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

\[
\begin{align*}
a & = 4 & b & = 5/2 \\
a^2 & = 16 & b^2 & = 25/4
\end{align*}
\]

\[ \frac{x^2}{16} + \frac{4y^2}{25} = 1 \] (4)

3. Find the center, vertices, the foci, and the equations of the asymptotes of the hyperbola. Sketch its graph.

\[
\frac{(y+2)^2}{9} - \frac{(x+2)^2}{4} = 1
\]

Vertical

\[
\begin{align*}
C & = (-2, -2) \\
V & = (-2, -2 \pm 3) \\
W & = (-2 \pm 2, -2) \\
c^2 & = a^2 + b^2 \\
c & = \sqrt{13} \\
F & = (-2, -2 \pm \sqrt{13}) \\
(y+2) & = \pm \frac{3}{2} (x+2)
\end{align*}
\]
4. Solve the system.
\[
\begin{cases}
y^2 - 4x^2 = 4 \\
9y^2 + 16x^2 = 140
\end{cases}
\]
\[
\begin{cases}
4y^2 - 16x^2 = 16 \\
9y^2 + 16x^2 = 140
\end{cases}
\]
\[
\begin{cases}
12 - 4x^2 = 4 \\
-4x^2 = -8
\end{cases}
\]
\[
\begin{cases}
y^2 = 12 \\
x^2 = 2
\end{cases}
\]
\[
\begin{cases}
y = \pm \sqrt{12} = \pm 2\sqrt{3} \\
x = \pm \sqrt{2}
\end{cases}
\]
4 points: \((\pm \sqrt{2}, \pm 2\sqrt{3})\)

5. **Mixing a silver alloy** A silversmith has two alloys, one containing 35% silver and the other 60% silver. How much of each should be melted and combined to obtain 100 grams of an alloy containing 50% silver?

Let \(x\) = amount of 35\% silver alloy

\[
\begin{cases}
35x + 60y = 50(100) \\
x + y = 100 \\
35x + 60y = 5000 \\
-35x + 35y = -3500
\end{cases}
\]
\[
25y = 1500 \Rightarrow y = 60g
\]

40g of 35\% silver alloy

60g of 60\% silver alloy

6a) Find the radian and degree measures of the central angle \(\theta\) subtended by an arc of 7 cm on a circle of radius 4 cm.

\[
S = r\theta \quad \theta = \frac{7\text{ cm}}{4\text{ cm}} = \frac{7}{4} \quad \text{radians}
\]
\[
\theta = \frac{7}{4} \left(\frac{180^\circ}{\pi}\right) = \left(\frac{315}{\pi}\right)^\circ \approx 100.27^\circ
\]

6a) \(\theta = \left(\frac{315}{\pi}\right)^\circ\)

6b) Find the area of the sector determined by \(\theta\) in part (a).

\[
A = \frac{1}{2}r^2\theta
\]
\[
A = \frac{1}{2}(4)^2\left(\frac{7}{4}\right) = 14 \text{ cm}^2
\]

6b) \(A = 14 \text{ cm}^2\)
7. A wheel of radius 5 inches is rotating at a rate of 40 rpm.
   a) Find the angular speed (in radians per minute).

\[
\omega = \frac{40 \text{ rev}}{\text{min}} \cdot \frac{2\pi \text{ radians}}{1 \text{ rev}} \\
\omega = 80 \pi \text{ rad/min}
\]

\(7a) \quad 80 \pi \frac{\text{ rad}}{\text{min}} \) (4)

b) Find the linear speed of a point on the circumference (in ft/min).

\[
V = r \omega
\]
\[
V = \left( \frac{5 \text{ in}}{1} \right) \left( \frac{80 \pi \text{ rad}}{\text{min}} \right) \left( \frac{1 \text{ ft}}{12 \text{ in}} \right) = \frac{100\pi}{3} \frac{\text{ ft}}{\text{min}}
\]

\(7b) \quad \frac{100\pi}{3} \frac{\text{ ft}}{\text{min}} \) (4)

8. Find the exact values of \( \sin \theta \), \( \cos \theta \) and \( \tan \theta \) for the acute angle \( \theta \) if \( \sec \theta = \frac{6}{5} \).

\[
\sec \theta = \frac{6}{5} \quad r^2 = x^2 + y^2 \quad y = \sqrt{11}
\]

Let \( x = 5 \), \( 36 = 25 + y^2 \)
\[
y^2 = 11 \quad y = \pm \sqrt{11}
\]
\[
r = 6 \quad \theta \text{ is an acute angle}
\]

\[
\sin \theta = \frac{\sqrt{11}}{6}, \quad \cos \theta = \frac{5}{6}
\]
\[
\tan \theta = \frac{y}{x}
\]

\(8) \quad \frac{\sqrt{11}}{6}, \quad \frac{5}{6}, \quad \frac{y}{x} \) (4)

9. Find the exact value.

\[
\csc(-2\pi/3) = \frac{1}{\sin \left( -\frac{2\pi}{3} \right)}
\]
\[
= \frac{1}{-\sin \frac{2\pi}{3}} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}
\]

\(9) \quad -\frac{2}{\sqrt{3}} \) (4)

10. Graph at least one complete period of \( y = 1 + \csc x \).

Shift the graph of \( y = \csc x \) up 1 unit.
11. Find the amplitude, period, phase shift, and graph at least one complete period for $y = -5\cos\left(\frac{1}{3}x + \frac{\pi}{6}\right)$.

\begin{align*}
0 &\leq \frac{1}{3}x + \frac{\pi}{6} \leq 2\pi \\
-\frac{\pi}{2} &\leq x \leq \frac{11\pi}{2} \\
-\frac{\pi}{2} &\leq x \leq \frac{11\pi}{2}, \text{ phase shift } x = -\frac{\pi}{2} \\
\text{Period} &= 6\pi \\
\text{Amplitude} &= 5 \\
y\text{-intercept } (0, -\frac{5\sqrt{3}}{2})
\end{align*}

12. Given the indicated parts of triangle $ABC$ with $\gamma = 90^\circ$, express the third part in terms of the first two.

\[ \beta, b, a \]

\[ \tan \beta = \frac{b}{a} \]

\[ a = \frac{b}{\tan \beta} \quad \text{or} \quad a = b \cot \beta \]

12. \[ a = b \cot \beta \]

13. A rocket is fired at sea level and climbs at a constant angle of $75^\circ$ through a distance of 10,000 feet. Approximate its altitude to the nearest foot.

\[ \sin 75^\circ = \frac{h}{10000} \]

\[ h = 10,000 \sin 75^\circ \text{ feet} \]

\[ h \approx 9659 \text{ feet} \]

14. Verify the identity. \textit{Show all work.}

\[ \tan^4 k - \sec^4 k = 1 - 2\sec^2 k \]

\[ \tan^4 k - \sec^4 k = (\tan^2 k - \sec^2 k)(\tan^2 k + \sec^2 k) \]

\[ = (-1)(\sec^2 k - 1 + \sec^2 k) \]

\[ = (-1)(2\sec^2 k - 1) \]

\[ = 1 - 2\sec^2 k \]
15. Find the **exact values** of the solutions of the equation that are in the interval \([0, 2\pi)\). 

\[
2\sin^2 u = 1 - \sin u \\
2\sin^2 u + \sin u - 1 = 0 \\
(2\sin u - 1)(\sin u + 1) = 0 \\
\sin u = \frac{1}{2}, \quad \sin u = -1 \\
u = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad u = \frac{3\pi}{2} \\
\]

16. If \(\cos \alpha = -\frac{2}{3}\) and \(\cos \beta = -\frac{1}{3}\) for third-quadrant angles \(\alpha\) and \(\beta\), find the **exact value** for:

\(\alpha\): 
\[x = -2, \quad y = -\sqrt{21}, \quad \beta: \quad x = -3, \quad y = -4, \quad r = 5, \quad r = 5\]

a) \(\sin(\alpha - \beta)\)
\[
\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \\
= \left(-\frac{2}{3}\right) \left(-\frac{1}{3}\right) - \left(-\frac{1}{3}\right) \left(-\frac{4}{5}\right) = \frac{3\sqrt{21} - 8}{25} \\
\]

b) \(\cos(\alpha - \beta)\)
\[
\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
= \left(-\frac{2}{3}\right) \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \left(-\frac{4}{5}\right) = \frac{6 + 4\sqrt{21}}{25} \\
\]

17. Given \(\sec \theta = -3\); \(90^\circ < \theta < 180^\circ\) find the **exact value** of \(\sin 2\theta\), \(\cos 2\theta\) and \(\tan 2\theta\). 

\(\cos \theta = -\frac{1}{3}\) 
\[x = -1, \quad y = \sqrt{8} = 2\sqrt{2}, \quad r = 3\]
\[
\sin 2\theta = 2\sin \theta \cos \theta = 2 \left(\frac{2\sqrt{2}}{3}\right) \left(-\frac{1}{3}\right) = -\frac{4\sqrt{2}}{9} \\
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(-\frac{1}{3}\right)^2 - \left(\frac{2\sqrt{2}}{3}\right)^2 = -\frac{7}{9} \\
\tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta} = \frac{2 \left(-2\sqrt{2}\right)}{1 - (2\sqrt{2})^2} \\
= \frac{-4\sqrt{2}}{-7} = \frac{4\sqrt{2}}{7} \\
\]

\[17a) \sin 2\theta : \frac{-4\sqrt{2}}{9} \quad (4) \]
\[17b) \cos 2\theta : \frac{-7}{9} \quad (4) \]
\[17c) \tan 2\theta : \frac{4\sqrt{2}}{7} \quad (4) \]
18. Find the **exact values** of the solutions of the equation that are in the interval \([0, 2\pi)\).

\[
\sin 2t + \sin t = 0 \\
2 \sin t \cos t + \sin t = 0 \\
\sin t (2 \cos t + 1) = 0 \\
\sin t = 0 \quad \cos t = -\frac{1}{2}
\]

\[\begin{align*}
t &= 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}
\end{align*}\]

19. Without using your calculator, find the **exact value** of the expression.

\[\csc [\cos^{-1}(-\frac{1}{4})]\]

Let \(\theta = \cos^{-1}(-\frac{1}{4})\)

\[
\begin{align*}
\cos \theta &= -\frac{1}{4} \\
x &= -1 \\
y &= \sqrt{15} \\
r &= 4 \\
\csc \theta &= \frac{r}{y}, \csc \theta = \frac{4}{\sqrt{15}}
\end{align*}\]

19. \[\frac{4}{\sqrt{15}}\]

20. Use inverse trigonometric functions to find the solutions of \(2 \tan^2 t + 9 \tan t + 3 = 0\) that are in \(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\), and approximate the solutions to four decimal places.

\[
\begin{align*}
2 \tan^2 t + 9 \tan t + 3 &= 0 \\
\tan t &= -\frac{9 \pm \sqrt{81-4(2)(3)}}{8} = -\frac{9 \pm \sqrt{57}}{4} \\
\tan t &= -\frac{9 + \sqrt{57}}{4}, \tan t = -\frac{9 - \sqrt{57}}{4} \\
t_R &\approx 0.3478 \\
t_R &\approx 1.3336 \\
t &\approx 0.3478 \\
t &\approx -1.3336
\end{align*}\]

20. \[-1.3336\]
21. In triangle $ABC$ if $\alpha = 42^\circ 10'$, $\gamma = 61^\circ 20'$ and $b = 19.7$ find the value of side $a$. \hspace{1cm} (8.1 \#5)

\[ \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} \]

\[ a = \left(19.7\right) \left(\sin 42^\circ 10'\right) \]

\[ \frac{a}{\sin 76^\circ 30'} \]

\[ a \approx 13.6 \]

\[ \beta = 180^\circ - (\alpha + \gamma) = 76^\circ 30' \]

\[ \beta \approx 13.6 \]

22. A triangular plot of land has sides of lengths 420 feet, 350 feet, and 180 feet. Approximate the smallest angle between the sides. \hspace{1cm} (8.2 \#18)

\[ \cos \beta = \frac{420^2 + 350^2 - 180^2}{2 \times 420 \times 350} \]

\[ \beta \approx 24.979^\circ \]

\[ \cos \beta \approx 0.90646 \]

\[ \beta \approx 25^\circ \]

22. \hspace{1cm} \approx 25^\circ \]

23. Approximate the area of a parallelogram that has sides of lengths $a$ and $b$ (in feet) if one angle at a vertex has measure $\theta$. \hspace{1cm} (8.2 \#43)

\[ a = 12.0 \text{ ft}, \ b = 16.0 \text{ ft}, \ \theta = 40^\circ \]

\[ A = 2 \left(\frac{1}{2} \times 12 \times 16 \times \sin 40^\circ\right) \]

\[ A \approx 123.4 \text{ ft}^2 \]

\[ A \approx 123.4 \text{ ft}^2 \]

23. \hspace{1cm} A \approx 123.4 \text{ ft}^2 \]

**Bonus:** Find all exact values for the solutions of the equation. \hspace{1cm} (7.2 \#17)

\[ \sin \left(2x - \frac{\pi}{3}\right) = \frac{1}{2} \]

Let $\theta = 2x - \frac{\pi}{3}$

\[ \sin \theta = \frac{1}{2} \]

\[ 2x - \frac{\pi}{3} = \frac{\pi}{6} + 2\pi n, \ 2x = \frac{5\pi}{6} + 2\pi n \]

$\theta = \frac{5\pi}{6} + 2\pi n$ \hspace{1cm} $x = \frac{\pi}{4} + \pi n$, $\frac{7\pi}{12} + \pi n$

\[ \theta = \frac{\pi}{6} + 2\pi n \]

\[ 2x = \frac{\pi}{2} + 2\pi n \]

\[ x = \frac{\pi}{4} + \pi n \]

Bonus: \hspace{1cm} $x = \frac{\pi}{4} + \pi n$, $\frac{7\pi}{12} + \pi n$