Math 554 Qualifying Exam

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You may use any theorems from the book. Other results you use must be proved. Make sure to double check your calculations and support your arguments.

1. Let

\[ A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}. \]

(a) (10) Find the invariant factors and Jordan canonical form of \( A \).

(b) (10) Find a diagonalizable matrix \( D \) and a nilpotent matrix \( N \) such that \( A = D + N \) and \( DN = ND \).

2. (15) Let \( T \) be a linear operator on the finite-dimensional space \( V \) over a field \( F \), let \( R \) be the range and \( N \) be the null space of \( T \). Prove that \( R \) has a \( T \)-invariant complement if and only if \( R \) and \( N \) are independent. (Note: Two subspaces \( U \) and \( W \) are independent if whenever \( u + w = 0 \) with \( u \in U \) and \( w \in W \), then \( u = w = 0 \).)

3. (20) Let \( T \) be a linear transformation on a finite dimensional vector space \( V \) over the field \( F \). Let \( W \) be a proper nontrivial subspace of \( V \). Show that \( \dim(TW) + \dim(N(T) \cap W) = \dim W \) where \( TW \) is the image of \( T \) on \( W \) and \( N(T) \) is the null space of \( T \).

4. Let \( T \) be a linear operator on a finite dimensional inner product space \( V \) over \( \mathbb{C} \). Let \( W \) be a \( T \)-invariant subspace of \( V \). Let \( W^\perp \) be the orthogonal complement of \( W \).

(a) (5) Show that \( W^\perp \) is \( T^* \)-invariant.

(b) (5) If \( T \) is normal and \( W \) is a span of some eigenvectors of \( T \), then \( W^\perp \) is both \( T \) and \( T^* \) invariant. (Note: \( T \) is normal if \( T^*T = TT^* \).)

(c) (10) If \( T \) is normal and \( W \) is \( T \)-invariant, show that \( W \) is also \( T^* \)-invariant.

(d) (5) Show that if \( T \) is normal and \( W \) is both \( T \) and \( T^* \) invariant, then \( T|_W \) is normal on \( W \).

5. (20) Let \( T \) be a linear operator on a finite dimensional inner product space \( V \) over \( \mathbb{C} \). Prove that \( T \) is self-adjoint if and only if \( \langle T\alpha|\alpha \rangle \) is real for every \( \alpha \) in \( V \). (Note: \( T \) is self-adjoint if \( T^* = T \).)