Math 553 Qualifying Exam

August, 2016
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You must provide necessary arguments to support your claims. You may use any theorems and basic examples from the textbooks but not from homework. Solutions must be shortened for simplicity.

1. (20) Show that a group of order $7007 = 7^2 \times 11 \times 13$ is abelian and find all isomorphism types of this group.

2. (20) Let $G$ be a finitely generated abelian group and $H$ be a subgroup. Let $G_t$ and $H_t$ be the corresponding torsion subgroups of $G$ and $H$, respectively. If $G/G_t$ has rank $n$ and $H/H_t$ has rank $k$, show that $(G/H)/(G/H)_t$ has rank $n - k$.

3. (20) Let $\mathbb{Z}[i]$ be the ring of Gaussian integers. Let $N(a + bi) = a^2 + b^2$ be the norm function on $\mathbb{Z}[i]$.
   (a) Show that $\mathbb{F} = \mathbb{Z}[i]/(2 - 3i)$ is a field, where $(2 - 3i)$ is the ideal generated by $2 - 3i$.
   (b) Determine the number of elements in $\mathbb{F}$ from part (a) and the characteristic $p$ of the field $\mathbb{F}$.

4. (20) Let $R$ be a PID. An ideal $P$ is said primary if whenever $ab \in P$ but $a \notin P$ imply $b^n \in P$ for some $n$. Show that a nontrivial ideal $P$ is primary if and only if for some $n$, $P = (p^n)$ where $p \in R$ is a prime element. (Don’t forget $R$ is a PID.)

6. (20) Let $\zeta_8 = \frac{\sqrt{2}}{2}(1 + i)$ be the primitive 8th root of unity.
   (a) Show that $\mathbb{Q}(\zeta_8, \sqrt{2}) = \mathbb{Q}(i, \sqrt{2})$ and the extension $\mathbb{Q}(\zeta_8, \sqrt{2})/\mathbb{Q}$ is a Galois extension.
   (b) Find the order of the Galois group $\text{Aut}_{\mathbb{Q}} \mathbb{Q}(\zeta_8, \sqrt{2})$.
   (c) Determine the Galois group $\text{Aut}_{\mathbb{Q}} \mathbb{Q}(\zeta_8, \sqrt{2})$. 