(1) a) Prove that if $G$ is a group of order $2^2 \cdot 5^2 \cdot 11$, then $G$ contains either a normal subgroup of order 11, or a normal subgroup of order $2^2 \cdot 5^2$.

b) Prove that all groups of order $2^2 \cdot 5^2 \cdot 11$ are solvable.

(2) Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ denote the cyclic group of order $n$, and for $a \in \mathbb{Z}$, let $\pi$ denote its image in $\mathbb{Z}_n$.
Consider the group
$$A = \mathbb{Z}_{3,5^2,7} \times \mathbb{Z}_{5^2,7}$$
and let $C \leq A$ be the cyclic subgroup generated by $(15,35)$.

a) Find the invariant factor and the elementary divisor decompositions of the quotient group $A/C$.
b) Do the elements $(7,35)$ and $(35,7)$ represent the same element in $A/C$? Explain your answer.

(3) Let $K \leq S_7$ be the subgroup generated by $\sigma = (1 \ 2)(3 \ 4)$ and $\tau = (1 \ 3)(2 \ 4)$. Describe the group $K$, and calculate the number of subgroups in $S_7$ that are conjugate to $K$.

(4) Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.

a) Show that the ideal of $R$ generated by $6 + \sqrt{-5}$ is maximal.
b) Let $I = (41)$ be the ideal of $R$ generated by $41 = 41 + 0\sqrt{-5}$. Show that the quotient ring $R/I$ is isomorphic to $\mathbb{Z}/41\mathbb{Z} \times \mathbb{Z}/41\mathbb{Z}$. (The Chinese Remainder Theorem may be helpful here.)

(5) Let $p(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n$, and let $G$ be the Galois group of $p(x)$. Prove that if $G$ is abelian, then $|G| = n$.

(6) Let $p(x) = x^6 - 3 \in \mathbb{Q}[x]$, and let $K \subset \mathbb{C}$ be the splitting field for $p(x)$.

a) Find a generating set for the $\text{Gal}(K/\mathbb{Q})$ (the Galois group of $p(x)$), and describe the action of each generator as a permutation of the roots of $p(x)$.
b) Calculate $\text{Gal}(K/\mathbb{Q})$ (it is isomorphic to a familiar group).
c) Prove that there is a unique intermediate subfield $\mathbb{Q} \subset E \subset K$ satisfying $[E : \mathbb{Q}] = 4$, and show that $E \cap \mathbb{R} = \mathbb{Q}(\sqrt{3})$. 