Problem 1 - Two values, \(x \) and \(y\) are selected at random and independently of each other from the interval \([0,1]\) with all the outcomes being equally likely. What is the probability that \(|x - y| > 1/2\)?

Problem 2 - Let \(B\) be a "mysterious blob". If you hold \(B\) under the sun and see a perfectly round shadow, \(B\) does not need to be a perfectly round ball. For example, it could be a cylinder that is held perfectly vertically. Suppose that you hold \(B\) under the sun and see a perfectly round shadow and then rotate \(B\) to a new orientation and again see a perfectly round shadow of the same diameter. Must \(B\) be a perfectly round ball? What if you are able to rotate \(B\) twice, getting now two new orientations, and all three shadows that you see are perfectly round and of the same diameter? Must \(B\) be a ball?

Problem 3 - Show that \(\sum_{n=1}^{\infty} \frac{1}{4^n + 16n + 23n^3 + 14n^3 + 9} = \frac{10 - n^2}{6}\).

Problem 4 - Let \(p_n(x)\) be a polynomial of degree \(n\) such that \(p(0) = 1, p(1) = a, p(2) = a^2, \ldots, p(n) = a^n\) for some number \(a\). What is \(p(n + 1)\)?

Problem 5 - Write 500 to 700 words (complete with references) on an application of mathematics to medical imaging.

Team Problem - Some containers of blood plasma have become contaminated in a laboratory, with pure samples having no contaminant and tainted samples having the same concentration, \(1\%\), of the contaminant. A medical scanning device can be used to detect the percentage concentration of this contaminant in a sample of plasma. It is very expensive to run, however, so the lab technician would like to minimize the number of trials run while still being certain which samples are tainted.

In a different universe, the technician could take 1 cubic millimeter from the first bag, 2 cubic millimeters from the second bag, 4 cubic millimeters from the third bag, 8 cubic millimeters from the fourth bag, and so on, and from the level of concentration of the contaminant (the binary expansion of \(k\) in the fraction of concentration \(k/(2^n - 1)\)), a single trial would suffice. In this universe, unfortunately, the technician is further restricted to using equal amounts of plasma from each bag tested in a given trial. It turns out that with four containers to test, \(A, B, C,\) and \(D\), the technician can test mixtures \(ABC, ABD,\) and \(ACD\) - and this will uniquely determine which containers are contaminated. If we let the level of contamination for \(ABC\) be \(c_1\%\), for \(ABD\) be \(c_2\%\), and for \(ACD\) be \(c_3\%\), then we can add these together to get \(3A + 2B + 2C + 2D = 3(c_1 + c_2 + c_3)\); if the right-hand side is even, then \(A\) must not be contaminated, while if the right-hand side is odd, then \(A\) is contaminated. From here it is a simple matter to solve for the status of ("resolve") containers \(B, C,\) and \(D\).

A. Find another non-isomorphic way to resolve 4 containers in 3 trials (non-isomorphic means a way that does not involve mixing 3 different samples in 3 different ways).
B. How can you resolve 7 containers in 5 trials?
C. How can you resolve 9 containers in 6 trials?
D. How can you resolve 11 containers (or more) in 7 trials?
E. How can you resolve 14 containers (or more) in 8 trials?
F. How few trials are required to resolve 100 containers? (The author of the problem does not know the answer, but he has a means of resolving 100 containers in 57 trials. See if you can do better. Note also that an answer of 75 is trivial - resolve 4 containers in 3 trials 25 times - and will therefore not be considered for even partial credit. Note also that if you solve part (E), you will have effectively found a way to resolve 100 containers in 58 trials by using the method for (E) 7 times and then testing the last two containers individually.)