Math 571 Qualifying Exam, August 2018.

Use a separate piece of paper for each problem.

1. Suppose $X = [0,3]$ and $\sim$ is the equivalence relation on X such that the equivalence classes consist of the set $[0,1)$ as well as the one-point sets $\{x\}$ for each $x \in [1,3]$. Is the quotient space $X/\sim$ Hausdorff?

2. Let (X,d) be a metric space and $f : X \to X$ be a continuous map such that for all $x \neq y \in X$ we have $d(f(x), f(y)) < d(x,y)$.

   (a) Give an example to show that if X is not compact, then f need not have any fixed points.
   (b) If X is compact, show that f has a fixed point x_0.
   (c) Show that if f has a fixed point, it is unique.

3. Give $\{0,1\}$ the discrete topology and let $X = \prod_{n\in\mathbb{N}}\{0,1\}$, with the product topology.

   (a) Find all connected components of X.
   (b) Find all limit points of X.

4. Let $p : \tilde{X} \to X$ be a covering map. Let Y be connected and $y_0 \in Y$. Let $f, g : Y \to \tilde{X}$ be continuous maps such that:

   (i) $f(y_0) = g(y_0)$, and
   (ii) $p \circ f = p \circ g$.

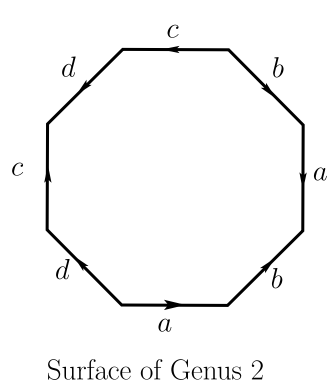
   Prove that $f = g$.

5. Prove that the torus $\mathbb{T}^2$ and the surface M_2 of genus two are not homeomorphic. Give plenty of details. By definition, M_2 is obtained by identifying the sides of an octagon as shown in the figure on the left side of the bottom of the page.

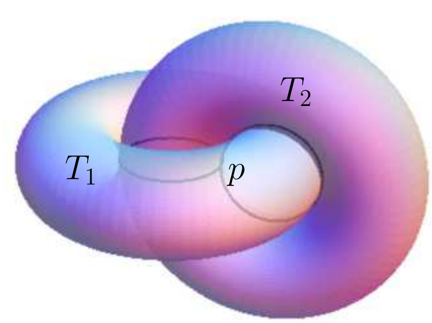
6. Let $T_1 \subset \mathbb{R}^3$ be the torus obtained by sweeping a circle of radius 1 orthogonally around the horizontal circle of radius 2 centered at the origin. Let T_2 be the torus obtained by shifting T_1 two units along a horizontal line, and then rotating it by $\pi/2$ radians around that same line. See the figure on the right side of the bottom of the page.

   Let S denote the union in $\mathbb{R}^3$ of T_1 and T_2. Notice that $T_1 \cap T_2$ is a wedge of two circles. Let p denote the point where these two circles meet.

   Compute $\pi_1(S,p)$ and express your answer in the simplest possible terms.



Surface of Genus 2
(Problem 5)



Two linked tori
(Problem 6)

Math 571 Qualifying Exam, January 2018.

Use a separate piece of paper for each problem.

Several of the problems refer to the following subspaces of $\mathbb{R}^2$:

$$\begin{aligned} S &= \{x \times y \in \mathbb{R}^2 \,:\, -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\} && \textit{(the square)}, \\ D &= \{x \times y \in \mathbb{R}^2 \,:\, x^2 + y^2 \leq 1\} && \textit{(the closed disc)}, \quad \text{and} \\ \mathbb{S}^1 &= \{x \times y \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\} && \textit{(the unit circle)}. \end{aligned}$$

1. Let A be a proper subset of X and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) \setminus (A \times B)$ is connected.

2. Let $f : X \to Y$ be a continuous function between topological spaces.

   (a) Prove that for any $A \subset X$ we have $f\left(\overline{A}\right) \subset \overline{f(A)}$.

   (b) Give an example for which $f\left(\overline{A}\right) \neq \overline{f(A)}$.

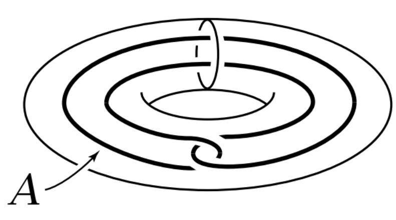
   (c) Add a (reasonable) additional hypothesis about X so that for any $A \subset X$ we have $f\left(\overline{A}\right) = \overline{f(A)}$. Prove it!

3. A topological space X is *very compact*[1] if X is compact and every $Y \subset X$ is also compact. Prove that if X has infinitely many points and X is very compact, then X is not Hausdorff.

4. Define an equivalence relation $\sim$ on the square S so that for all $s \in [-1, 1]$ we have $s \times -1 \sim s \times 1$, for all $t \in [-1, 1]$ we have $-1 \times t \sim 1 \times t$ and (of course) so that for all $s \times t \in S$, $s \times t \sim s \times t$.

   Let $T = S/\sim$ be the quotient space. Prove that T is homeomorphic to the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

5. Let $X = \mathbb{S}^1 \times D$ (solid torus) and let $A \subset X$ be the circle shown in the picture below. Prove that there is no retraction from X onto A.



6. Let X be the quotient space obtained from the disjoint union of D and $\mathbb{S}^1$ by identifying (gluing) each point at angle θ on $\partial D \cong \mathbb{S}^1$ to the point at angle 3θ in $\mathbb{S}^1$. Compute $\pi_1(X, x_0)$ for some choice of basepoint x_0.

[1] not an official term

QUALIFYING EXAMINATION
MATH 571
AUGUST 2017

1. Consider the following five subspaces of the plane:

$$0, \quad 2, \quad 4, \quad 6, \quad 8,$$

where for example we take 4 to be the union of four segments and 8 to be two circles joined at a point. Use connectedness to explain why no pair are homeomorphic. Which are homotopy equivalent?

2. Consider the set $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \ldots$ of all sequences of reals, endowed with the product topology, and the subset B of all bounded sequences of reals. Is B open, closed, both, or neither? What if $\mathbb{R}^\omega$ is instead given the box topology? In each case, justify your answer.

3. a) Show that every order topology is Hausdorff.
b) Give an example of an order on a countable set for which the order topology is not discrete.

4. Prove that every closed subset of a compact topological space is compact.

5. Define what it means for the topological space X to be contractible, and what it means for Y to be a retract of X. Show that if X is contractible and Y is a retract of X then Y is also contractible.

6. Let X be the twice-punctured unit sphere $S^2 \setminus \{(0,0,1),(0,0,-1)\}$. Consider the equivalence relation $\sim$ on X identifying the points on each great circle through the missing poles. Describe the obtained quotient space $Y = X/\sim\, = q_Y(X)$. Describe a finer equivalence relation $\approx$ on X and its quotient space $Z = X/\approx\, = q_Z(X)$ so that Z is a 2-fold covering space of Y, $p : Z \longrightarrow Y$, with $q_Y = p \circ q_Z$, and describe the covering map p.

7. Let X be the complement of the nonnegative coordinate axes in $\mathbb{R}^3$. Find the fundamental group $\pi_1(X)$. (Hint: Use a deformation retract.)

QUALIFYING EXAMINATION
MATH 571
JANUARY 2017

1. Consider X, the union of two spheres of unit radius in $\mathbb{R}^3$ joined at a point. Describe five (pairwise) nonhomeomorphic intersections of X with a plane, and give topological properties distinguishing them.

2. Prove or disprove: If C and D are distinct components of a topological space X then there is a separation $X = A \cup B$ with $C \subset A$ and $D \subset B$.

3. Prove that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

4. Prove that if X is a locally compact noncompact Hausdorff space and Y is its one-point compactification then Y is also Hausdorff.

5. Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x^2 + y^2 + z^2 \leq 1\}$ be the punctured closed unit ball in $\mathbb{R}^3$ and $A = \{(x, y, 0) \in \mathbb{R}^3 \mid 0 < x^2 + y^2 \leq 1\} \subset X$ be its equatorial punctured disk. Is there a retraction from X onto A? What if X is replaced by the closed unit ball with the points on the z-axis removed, $X' = X \setminus \{(0, 0, z) \in \mathbb{R}^3 \mid |z| \leq 1\}$? In each case, describe a retraction or prove that none exists.

6. If E and B are topological spaces, define what it means for the continuous surjection $p : E \longrightarrow B$ to be a covering map. Prove that the product $(p, p') : E \times E' \longrightarrow B \times B'$ of covering maps $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ is a covering map.

7. Use Seifert-van Kampen to find the fundamental group of the union X of two horizontal planes in $\mathbb{R}^3$ and three vertical segments connecting them (taking as basepoint x_0 the midpoint of one of the segments).

QUALIFYING EXAMINATION
MATH 571
JANUARY 2016

1. If the integers are given the cofinite (finite complement) topology, what are the compact connected subsets? Prove your answer.

2. Let the cross $X = \{(x, y) \in [-1, 1]^2 \mid x = 0 \text{ or } y = 0\}$ be the subspace of the Euclidean plane consisting of the coordinate axes in the square, with the metric it inherits from the plane. Describe six open balls in X, no two of which are homeomorphic, and explain why not.

3. Let S_r be the circle of radius r centered at $(r, 0)$ in the plane. Give the countable union of circles meeting at the origin $X = \cup_{n=1}^{\infty} (S_n \cup S_{1/(n+1)})$ the topology it inherits from the plane. Define bijections $f, g : X \to X$ fixing the origin and mapping each circle to another circle along rays from the origin by: f takes each circle to the next larger one, and g switches each S_n with $S_{1/(n+1)}$. Prove that one of f, g fails to be a homeomorphism.

4. Prove that the topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X | x \in X\}$ is closed in the product $X \times X$.

5. Give an example, or say that there is none, of a subset of the closed unit disk D in the plane, containing the unit circle S^1, for which S^1 is:
a) not a retract;
b) a retract but not a deformation retract;
c) a deformation retract but not homotopy equivalent;
d) homotopy equivalent but not homeomorphic.

6. Let $p : E \longrightarrow B$ be a covering map, with $p(e_0) = b_0$. The lifting correspondence $\varphi : \pi_1(B, b_0) \to p^{-1}(b_0)$ takes each loop class $[f]$ to the endpoint $\tilde{f}(1)$ of the lifted path $\tilde{f}$ starting at $\tilde{f}(0) = e_0$. If E is simply connected, show that φ is a bijection.

7. Recall that the projective plane $\mathbb{P}^2$ is the quotient of the unit sphere by the equivalence relation identifying antipodal points. Use Seifert-van Kampen (once) to compute the fundamental group $\pi_1(X, x_0)$ of two copies of the projective plane joined at a point x_0. (You needn't rederive $\pi_1(\mathbb{P}^2, x_0)$.)

QUALIFYING EXAMINATION
MATH 571
AUGUST 2016

1. Endow the nonnegative reals $X = [0, \infty)$ with the topology consisting of the sets (a, ∞) for $a \geq 0$, in addition to X and $\emptyset$. What are the compact subsets of this space? Prove your answer.

2. Prove that the continuous image of a connected topological space is connected.

3. Divide the first ten Roman numerals

$$I, II, III, IV, V, VI, VII, VIII, IX, X$$

(where e.g. IV is taken to be the union of seven closed segments, including four short horizontal ones, in the plane) into homeomorphism classes. Justify your answer.

4. Define the one-point compactification X^* of a locally compact Hausdorff space X. Prove that if X and Y are locally compact Hausdorff spaces and the continuous map $f : X \longrightarrow Y$ is proper, i.e. the pre-image of every compact set is compact, then f can be extended to a continuous map $f^* : X^* \longrightarrow Y^*$.

5. Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^3$ be the solid cylinder and $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \subset X$ be its boundary cylinder. Prove that there is no retraction of X onto A.

6. Describe three 3-fold covering spaces E_1, E_2, and E_3 of the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, no pair of which are homeomorphic. Give the corresponding covering maps.

7. Let X be the union of the nonnegative axes in $\mathbb{R}^3$ and the sphere S^2 centered at $(1, 1, 1)$ which is tangent to them. Compute the fundamental group of X using Seifert-van Kampen.

1. Let $(Y, \mathcal{T})$ be a topological space, and let X be a set such that there exists a surjective function $f : X \longrightarrow Y$.

Let $\mathcal{T}_1$ be the smallest topology in X that makes f a quotient map; $\mathcal{T}_2$ the smallest topology in X that makes f continuous; $\mathcal{T}_3$ the smallest topology in X that makes f an open map; $\mathcal{T}_4$ the smallest topology in X that makes f an closed map; $\mathcal{T}_5$ the smallest topology in X that makes f an open and closed map; and $\mathcal{T}_6$ the smallest topology in X that makes f a closed and continuous map.

Compare four pairs of topologies from among $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6\}$ (your choices), and justify your answers by examples if needed.

2.

a) Show that a topological space A is locally connected if and only if each connected component of A is open in A.

b) Let X and Y be topological spaces such that X is locally connected. Let $f : X \longrightarrow Y$ be a continuous, surjective, closed map (taking closed sets to closed sets). Show that Y is locally conected.

3. Let X be a compact Hausdorff space, and let A and B be disjoint closed nonempty sets of X. Show that there exist disjoint open sets K and W such that $A \subset K$ and $B \subset W$.

4. Let C be the boundary of the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ (i.e., the union of the six faces). Label a, b, c, d the four edges of the top face, oriented clockwise. Identify a with $-c$, and b with $-d$, and call $\widetilde{C}$ the resulting quotient space.

Use the van Kampen Theorem to compute the fundamental group of $\widetilde{C}$.

5. Suppose $f : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{T}^2$ is continuous (for $\mathbb{P}^2$ the projective plane, and $\mathbb{T}^2$ the torus).

a) Show that $f_* : \pi_1(\mathbb{P}^2 \times \mathbb{P}^2) \longrightarrow \pi_1(\mathbb{T}^2)$ is the trivial homomorphism.

b) Show that f is null-homotopic; that is, it is homotopic to a constant map (you may use the general lifting property).

6. Let E and Y be Hausdorff, connected and locally path connected, so that E is compact and simply connected. Suppose there is a covering map $p : E \longrightarrow Y$. Show that for any base point $b \in Y$, we have that $\pi_1(Y, b)$ is a finite group.

QUALIFYING EXAMINATION
MATH 571
JANUARY 2014

1. Let $p : X \longrightarrow Y$ be continuous, and suppose that $f : Y \longrightarrow X$ is continuous, with $p \circ f = i_Y$, the identity on Y. Show that p is a quotient map.

2. Is the unit square $I \times I$ compact in the topology it inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology? Is it locally compact? Justify your answers.

3. Prove that a compact set in a Hausdorff space is closed.

4. Describe the one-point compactification Y of the countable union of lines $X = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 \in \mathbb{N}\}$. Is Y connected? Locally connected? Path connected? Locally path connected? Simply connected? Explain.

5. Prove that there is no retraction from the solid torus onto the torus. What curve can be removed from the solid torus to enable such a retraction? Describe the retraction.

6. Use the universal cover to prove that if γ is a loop in the projective plane then $\gamma * \gamma$ is nullhomotopic.

7. Use Seifert-van Kampen to compute the fundamental group of the union of a sphere and one of its diameters.

QUALIFYING EXAMINATION
MATH 571
AUGUST 2014

1. Let $f, g : X \longrightarrow Y$ be continuous maps, where X is a topological space and Y is a Hausdorff space. Prove that the set $S = \{x \in X | f(x) = g(x)\}$ is closed.

2. Let X and Y be topological spaces and let $p : X \longrightarrow Y$ be a quotient map, and suppose that each fiber $p^{-1}(y)$ is connected and Y is connected. Prove that X is connected.

3. Prove that if X is a topological space and Y is a compact space then the projection $\pi_1 : X \times Y \longrightarrow X$ is a closed map. (You may use the Tube Lemma without proving it.)

4. In the plane, let X be the union of the coordinate axes and the line $x + y = 1$, and let Y be the union of X and the positive quadrant. Prove that there is no retraction of Y onto X.

5. Draw a subset Z of the plane homeomorphic to the one-point compactification of Y from the preceding problem. How many connected components does the complement of Z have, and how many of these are bounded?

6. Let E and B be topological spaces and let $p : E \longrightarrow B$ be a two-sheeted cover. Describe a non-identity homeomorphism $h : E \longrightarrow E$ satisfying $h \circ h = id$ and prove that h is a homeomorphism.

7. Consider the space X consisting of a sphere glued, along its equator, into the hole of a torus. Use Seifert-van Kampen to compute the fundamental group of X.

Math 571 Topology Qualifying Exam January 2, 2013.
Olguta Buse

1. (18 points) Let $g : X \longrightarrow Y$ be a surjective function between two topological spaces.

a) Show that if the function g satisfies the condition

$$\text{For any subset } A \subset X, \text{ we have that } \mathrm{int}(g(A)) \subset g(\mathrm{int}(A)),$$

then g is continuous.

b) Is the reverse implication in part a) true?

c) Does the result in part a) still hold if g is not surjective?

2. (12 points) Let $p : E \longrightarrow B$ be a covering map, such that B is path connected and such that there exists a point $b_0 \in B$ with cardinality of $p^{-1}(b_0)$ equal to n.

a) Show that the sets $p^{-1}(b)$ have cardinality n, for all $b \in B$.

b) Show that if B is compact then E is also compact.

3. (12 points) Show that if $f : X \longrightarrow Y$ is a continuous map between a compact space X and a Hausdorff space Y, then the image $f(X)$ is homeomorphic to a quotient space of X.

4. (14 points) Let $X = \partial([0,1] \times [0,1] \times [0,1])$ be the hollow cube with the standard induced topology from $\mathbb{R}^3$. Let A be the upper face of the cube X consisting of all points of X of the form $(x, y, 1)$ and let B be the lower face of the cube consisting of all points of the form $(x, y, 0)$. We identify the faces A and B by identifying pairs of points $(x, y, 1)$ and $(x, y, 0)$, for all $0 \leq x, y \leq 1$, and denote by C the quotient space of X obtained this way.

Compute, with justifications, the fundamental group of C.

5. (12 points) Let $\Delta \subset S^n \times S^n$ be the diagonal in the product of the two n-dimensional unit spheres. Show that the map $f : (S^n \times S^n \setminus \Delta) \longrightarrow S^n$ given by $f(x, y) = x$ is a homotopy equivalence.

6. (14 points) Let (X, d_X) and (Y, d_Y) be non-empty metric spaces, and assume $\pi : X \longrightarrow Y$ satisfies

$$\pi(D_X(x, r)) = D_Y(\pi(x), r)$$

for all $x \in X$ and $r > 0$, where D denotes closed balls of radius r around the given points.

a) Show that π is surjective, continuous, and open.

b) Let $y_1, y_2 \in Y$, and suppose that $x_1 \in X$ is such that $\pi(x_1) = y_1$. Show that there is $x_2 \in X$ such that $\pi(x_2) = y_2$ and $\mathrm{d}_X(x_1, x_2) = \mathrm{d}_Y(y_1, y_2)$.

7. (12 points) Let $X = S^2 \vee S^2$ be the one point union of two 2-spheres joined at a point x_0, and $Y = S^1 \vee S^1$ its subspace consisting of the one points, union of two great diameters, also joined at x_0. Is there a retract $r : X \longrightarrow Y$? Justify your answer.

TOPOLOGY — SUMMER 2013

1) Let $f : E \longrightarrow B$ be a covering of a compact Hausdorff space B. Prove that E is compact if and only if each fiber $f^{-1}(b)$ $(b \in B)$ is finite.
2) Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$, and that $\mathbb{R}^2$ is not homemorphic to $\mathbb{R}^3$.
3) Let A and B be disjoint compact sets in a Hausdorff space X. Show that there exist disjoint open sets $U, V \subset X$ containing A and B respectively.
4) Define the one point compactification of a locally compact Hausdorff space X. Prove or disprove by means of a counter-example the following statement: Two locally compact Hausdorff spaces X, Y are homeomorphic whenever their one point compactifications $\overline{X}, \overline{Y}$ are homeomorphic.
5) Recall that a space is contractible if it is homotopically equivalent to a point. Show that if Y is contractible then
   (a) $X \times Y$ is homotopically equivalent to X.
   (b) any two maps from X to Y are homotopic.
6) Compute the fundamental group of
   (a) The plane $\mathbb{R}^2$ minus the set consisting of the letters T E S T.
   (b) The sphere S^2 minus the set consisting of the letters T E S T.
7) Give the definition of a quotient map. Prove that if $p : X \longrightarrow Y$ is a quotient map, and $g : X \longrightarrow Z$ is a continuous map that is constant on each set $p^{-1}(y)$, for $y \in Y$, then g induces a continuous map $f : Y \longrightarrow Z$ such that $f \circ p = g$.

Math 57100 IUPUI Topology Qualifying Exam August 9, 2012.
Olguta Buse

1. (10 points) Let X be a compact topological space and A a subset of X.

   (a) Define what is an accumulation point (or limit point) of A in X.

   (b) Show that if A is an infinite subset of X then A must have at least one accumulation point in X.

2. (10 points) Show that $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

   (Here S^m is the unit sphere in $\mathbb{R}^{m+1}$).

3. (10 points) Let X be a compact Hausdorff space and $f : X \longrightarrow Y$ a quotient map.

   Show that the following three statements are equivalent:

   (a) f is a *closed* map (i.e. it takes closed sets to closed sets.)

   (b) Y is a Hausdorff space.

   (c) The set $\Gamma_f = \{(x_1, x_2) \in X \times X \,|\, f(x_1) = f(x_2)\}$ is closed in $X \times X$.

4. (10 points) Consider the subspace $Q = \bigcup_{i \geq 0} Q_i$ of $\mathbb{R}^2$, where $Q_0 = \{(0, y) \,|\, 0 < y \leq 1\}$ and $Q_i = \big\{(\frac{1-y}{i}, y) \,|\, 0 < y \leq 1\big\}$. Show that Q is connected but not locally connected.

5. (10 points) On a square white piece of paper write in black ink the word TOPOLOGY and denote by V the subspace consisting of the remaining white part.

   (a) How many connected components does V have?

   (b) For every connected component of V pick a point x_0 and compute $\pi_1(V, x_0)$. Justify all possible answers.

6. (10 points) Let X and $\widetilde{X}$ be two spaces that are path connected, locally path connected and such that $\pi_1(X) = \mathbb{Z}/23\mathbb{Z}$, and consider a covering map $p : \widetilde{X} \longrightarrow X$. Show that if $\widetilde{X}$ is not the universal cover, then p is a homeomorphism.

7. (10 points)Consider the bouquet of circles $B = \bigvee_{i=1}^{i=15} S_i^1$ as a canonical subset of the torus $T^{15} = \prod_{i=1}^{i=15} S^1$. Can B be a retract of T^{15}? Justify your answer.

Math 57100 – Qualifying Examination – August 2011

1. Let $Y = \{z \in \mathbb{C} : z^3 \in [0,1]\}$ (a triod) be a subspace of the complex plane $\mathbb{C}$, with the metric inherited from $\mathbb{C}$. Find 5 pairwise non-homeomorphic open balls in Y. Explain why they are not homeomorphic.

2. Prove that $\mathbb{R} \times S^1$ is homeomorphic with $\mathbb{R}^2 \setminus \{(0,0)\}$.

3. In a metric space (X,d) for nonempty sets $A, B \subset X$ define

$$\mathrm{dist}(A,B) = \inf\{d(a,b) : a \in A, \quad b \in B\}.$$

Prove that if A is closed, B compact and $A \cap B = \emptyset$ then $\mathrm{dist}(A,B) > 0$.

4. Let X be a compact space, U its open subset, and $f : U \to [0,1]$ a continuous map. Prove that the set

$$\{(x,t) : x \in U, 0 \le t \le f(x)\} \cup \big((X \setminus U) \times [0,1]\big) \subset X \times [0,1]$$

is compact.

5. Let A, B be subsets of $[0,1]$. Let $X \subset \mathbb{R}^2$ be the union of closed segments joining the point $(0,1)$ with all points of $A \times \{0\}$ and closed segments joining the point $(0,-1)$ with all points of $B \times \{0\}$. Prove that X is connected if and only if $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.

6. Let X be a topological space and let $S^2 \subset \mathbb{R}^3$ be the unit 2-sphere with the metric d inherited from $\mathbb{R}^3$. Show that if $f, g : X \to S^2$ are continuous maps such that $d(f(x), g(x)) < 2$ for all $x \in X$ then f and g are homotopic.

TOPOLOGY QUALIFYING EXAM

OLGUTA BUSE

(1) Let $X = [0,3]/(1,2)$ be the quotient space of the interval $[0,3]$ in which all points of the interval $(1,2)$ are identified. Prove that

  (a) X is connected.

  (b) X is compact.

  (c) X is not Hausdorff.

(2) Which one of the following are true? If true, prove it. If false, give a counterexample.

  (a) If A is connected, then $\overline{A}$ is connected.

  (b) If A is path-connected, then $\overline{A}$ is path-connected.

(3) Show that a space X is simply connected if and only if any two continuous maps $f, g : S^1 \longrightarrow X$ are homotopic to each other.

(4) Consider two different points x_0, x_1 in the sphere S^2. Compute the fundamental group of the quotient space $S^2/x_0 \sim x_1$, where the two points are identified.

(5) Consider a connected and locally path-connected space X such that $\pi_1(X)$ is finite. Show that every continuous map $f : X \longrightarrow S^1$ is homotopic to the constant map.

(6) Prove that there is no retraction from the sphere S^2 onto its equator E.

(7) Let $f : X \longrightarrow Y$ be a covering map.

  (a) Prove that f is open.

  (b) Prove that if Y is compact and f is an n-fold covering with finite n, then X is compact.

Math 571 Topology Qualifying Exam January 3, 2008.
Olguta Buse

1.(12) pts) a) Show that any group homomorphism from $\pi_1(S^1)$ into $\pi_1(S^1)$ can be obtained as the homomorpism h_* induced by some continuous map $h : S^1 \to S^1$.

b) Show that there are infinitely many nonhomotopic retracts from $S^1 \vee S^1$ to S^1.

2.(10 points) Let X be a locally compact Hausdorff space and let $C(X,Y)$ be the set of continous functions with the compact open topology. Define the evaluation map $e : X \times C(X,Y) \to Y$ by $e(x,f) = f(x)$. Show that the evaluation map is continuous.

3. (10 points) An accurate map of Indiana is laid out flat on a table in a classroom at IUPUI. Prove that there is exactly one point on the map lying directly over the point which it represents. You may use only definitions and theorems that are proved in your textbook for this problem.

4.(10 points) Let $X = C_1 \cup C_2$ be the space obtained as the union of two great circles C_1 and C_2 of S^2 that are both passing through the north and south pole. Use Van Kampen's Theorem to compute $\pi_1(X)$.

5. (12 points) Consider X a topological space. Are the following statements true? Provide a proof or a counterexample depending on your answer:

a) If $A \subset X$ is connected, then any B with $A \subset B \subset \overline{A}$ is connected.

b) If X is connected, then $Int(X)$ is connected.

6. (12 points) Let A be a path connected subspace of a path connected topological space X which has a universal cover $p : \tilde{X} \to X$. Show that the following are equivalent:

a) If $i : A \to X$ is the inclusion map, then $i_* : \pi_1(A) \to \pi_1(X)$ is surjective.

b) The inverse image $p^{-1}(A)$ is path connected.

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1. (10 points) Let $\{Y_i\}_{i\in I}$ be a collection of topological spaces, and let $A_i \subset Y_i$. Show that $\Pi_{i\in I}A_i$ is dense in $\Pi_{i\in I}Y_i$ if and only if A_i is dense in Y_i for all $i \in I$.

2. (13 points) Let $p : E \longrightarrow B$ be a continuous map between two topological spaces. A (continuous) map $s : B \longrightarrow E$ is called a (continuous) section of p if $p \circ s = id_B$.

   (a) Prove that if p has a continuous section then p is a quotient map, (i.e. B has a quotient topology induced by p). Under these conditions show that if E is Hausdorff then B is Hausdorff as well.

   (b) Show that if p has a continuous section s and E is Hausdorff, then $s : B \longrightarrow E$ is a closed topological embedding; i.e., it is closed and a homeomorphism on the image.

3. (10 points) Let X be the topological space obtained by removing 5 points from the sphere S^2. Find $\pi_1(X)$.

4. (16 points) Let B and Y be path connected, locally path connected and semi-locally simply connected topological spaces whose fundamental groups are $\pi_1(B, b_0) = \mathbb{Z}/15\mathbb{Z}$ and $\pi_1(Y, y_0) = \mathbb{Z}/7\mathbb{Z}$.

   (a) Construct a concrete example of two such spaces B and Y.

   (b) How many (up to equivalence) nontrivial covering spaces are there of B? Explain.

   (c) Can an arbitrary continuous map $Y \longrightarrow B$ be lifted to all covering spaces of B. Justify.

   (d) Is there such a space Y which is a of covering B? Explain.

5. (12 points) Let $Y = \mathbb{C} \setminus \{-1, 1\}$, the complex plane with two points removed. Let $X = Y \cup_f D^2$, obtained by attaching a two-dimensional disc along $f : S^1 \longrightarrow Y$, where f is given by

$$f(e^{2\pi it}) = \begin{cases} -1 + e^{8\pi it} & 0 \le t \le 1/4 \\ 1 - e^{8\pi it} & 1/4 \le t \le 1/2 \\ -1 + e^{-8\pi it} & 1/2 \le t \le 3/4 \\ 1 - e^{-24\pi it} & 3/4 \le t \le 1. \end{cases}$$

   Find $\pi_1(X)$.

6. (10 points) Let X and Y be two locally compact Hausdorff topological spaces.

   (a) Define the one-point compactification of X.

   (b) A function $f : X \longrightarrow Y$ is proper if it is continuous and the preimages of compact sets are compact. Prove that $f : X \longrightarrow Y$ is proper if and only if it extends to a continous map between the one-point compactifications of X and Y.

7. (10 points) Define what it means for X and Y to have the same *homotopy type*. A space is contractible if it is homotopy equivalent to the point space. Prove that X is contractible if and only if the identity map 1_X is homotopic to a constant map $r : X \longrightarrow X$ (whose image is a single point).

QUALIFYING EXAMINATION MATH 571 AUGUST 2007

1. Prove or disprove: A topological space X is connected if and only if every continuous map from X into $\mathbb{Q} \subset \mathbb{R}$ is constant.

2. If X is a compact space, Y is a Hausdorff space, and $f : X \longrightarrow Y$ is a continuous bijection, prove that f is a homeomorphism. If X is not assumed to be compact, give a counterexample.

3. Let X be the one-point compactification of the integers $\mathbb{Z}$. Construct an embedding of X into the reals $\mathbb{R}$.

4. In $\mathbb{R}^2$, let $C = \{|x| = 2\}$ and $A = \{1 < |x| < 3\}$. Let X be a path connected set such that $C \subset X \subset A$. Show that $\pi_1(X)$ contains a subgroup isomorphic to $\mathbb{Z}$.

5. Let X be the infinite cone over T^3, i.e. $(\mathbb{R}_+ \cup \{0\}) \times T^3$ modulo the usual equivalence relation ($\{0\} \times T^3$ is collapsed to a point). Verify that X is not homeomorphic to $\mathbb{R}^4$.

6. Compute the fundamental group of the Mercedes logo using the Seifert-van Kampen Theorem.

7. Let $p : E \longrightarrow B$ be a covering map such that $p^{-1}(b)$ consists of two points for every $b \in B$. Show that there is a continuous map $f : E \longrightarrow E$ (other than the identity map) such that $p \circ f = p$.

QUALIFYING EXAMINATION MATH 571 AUGUST 2005

1. Let $H = [-1, 1] \times \{0\}$ and $V = \{0\} \times [-1, 0)$ in the plane and let $T = H \cup V$. Show T is not homeomorphic to the unit interval $I = [0, 1]$.

2. Let (X, d) be a metric space and let $\mathcal{F}$ be the family of all nonempty closed, bounded subsets of X. For $F, G \in \mathcal{F}$, define $D(F, G) = \inf\{r | F \subset G_r \text{ and } G \subset F_r\}$, where $F_r = \bigcup_{x \in F} B(x, r)$ and $B(x, r)$ is the open ball of radius $r > 0$ centered at x. Show that D is a metric on $\mathcal{F}$.

3. Prove that if X and Y are connected spaces then $X \times Y$ is connected.

4.Prove that every compact regular space is normal.

5. Describe the one-point compactification of a locally compact Hausdorff space and prove that it is compact.

6. A set X in $\mathbb{R}^n$ is star convex if there is $x_0 \in X$ such that for every $x \in X$, the segment from x_0 to x lies in X. If X is star convex, prove that X is simply connected.

7.Let $p : E \longrightarrow B$ be a covering map. Prove that if E is compact, then $p^{-1}(b)$ is finite for every $b \in B$.