Problem 1. A real number $x$ on $[0,1]$ is called a Liouville type number, if its decimal representation has the form

$$x = \sum_{j=1}^{\infty} \frac{1}{10^{n_j}},$$

where $n_j \in \mathbb{N}$ are such that

$$n_{j+1} - n_j \geq j!, \quad j = 1, 2, \ldots.$$ (Liouville proved that all such numbers are transcendental.) Prove that the set of Liouville type numbers is uncountable.

Problem 2. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{|\sin(nx)|}}$$

converges almost everywhere on $[0, 2\pi]$.

Problem 3. Let $f(x)$ be an absolutely continuous function on an interval $[a, b]$ such that $f'(x) \in L^3[a, b]$. Prove that there exists a constant $C > 0$ such that

$$|f(y) - f(x)| \leq C|y - x|^{2/3}$$

for all $a \leq x < y \leq b$.

Problem 4. Let $f(x) \in L^1(-\infty, \infty)$. Prove that

$$\lim_{a \to \infty} \int_{-\infty}^{\infty} \frac{f(x) \, dx}{1 + (x-a)^2} = 0.$$

Problem 5. Let $f \in L^6[0, 1]$, $g \in L^7[0, 1]$, and $h \in L^8[0, 1]$. Prove that $fgh \in L^2[0, 1]$.

Problem 6. Let $E_1, E_2, \ldots$ be a sequence of measurable sets on the line such that $m(E_j \triangle E_k) < \infty$ for all $j, k = 1, 2, \ldots$ and

$$\lim_{j,k \to \infty} m(E_j \triangle E_k) = 0, \quad A \triangle B := (A \setminus B) \cup (B \setminus A).$$

Prove that there exists a measurable set $E$ such that

$$\lim_{j \to \infty} m(E_j \triangle E) = 0.$$
Problem 1. A real number is called transcendental if it is not a root of a polynomial with integer coefficients. Let $E$ be the set of transcendental numbers on the interval $[0, 1]$. Prove that the set $E$ is measurable and $mE = 1$.

Problem 2. Let $A$ and $E$ be disjoint sets on the line and let $E$ be measurable. Prove that

$$m^*(A \cup E) = m^* A + mE.$$ 

Problem 3. Prove that for any integrable function $f$ on the interval $[a, b]$,

$$\lim_{n \to \infty} \int_a^b f(x) |\sin(nx)| \, dx = \frac{2}{\pi} \int_a^b f(x) \, dx.$$ 

Problem 4. Let $f(x) \in L^1(-\infty, \infty)$. Prove that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x - n) \arctan \left( \frac{x}{n} \right) \, dx = \frac{\pi}{4} \int_{-\infty}^{\infty} f(x) \, dx.$$ 

Problem 5. Let $f(x)$ be a nonnegative integrable function on the interval $[a, b]$. Prove that

$$\left( \int_a^b f(x) \sin x \, dx \right)^2 + \left( \int_a^b f(x) \cos x \, dx \right)^2 \leq \left( \int_a^b f(x) \, dx \right)^2.$$ 

Problem 6. Let $f(x) \in L^2[0, 1]$. Define

$$g(x) = \frac{1}{x^{4/3}} \int_0^x f(t) \, dt.$$ 

Prove that $\|g\|_1 \leq 6\|f\|_2$. 
1. Let 
\[ A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}. \]

(a) (10) Find the invariant factors and Jordan canonical form of \( A \).

(b) (10) Find a diagonalizable matrix \( D \) and a nilpotent matrix \( N \) such that \( A = D + N \) and \( DN = ND \).

2. (15) Let \( T \) be a linear operator on the finite-dimensional space \( V \) over a field \( F \), let \( R \) be the range and \( N \) be the null space of \( T \). Prove that \( R \) has a \( T \)-invariant complement if and only if \( R \) and \( N \) are independent. (Note: Two subspaces \( U \) and \( W \) are independent if whenever \( u + w = 0 \) with \( u \in U \) and \( w \in W \), then \( u = w = 0 \).)

3. (20) Let \( T \) be a linear transformation on a finite dimensional vector space \( V \) over the field \( F \). Let \( W \) be a proper nontrivial subspace of \( V \). Show that \( \dim(TW) + \dim(N(T) \cap W) = \dim W \) where \( TW \) is the image of \( T \) on \( W \) and \( N(T) \) is the null space of \( T \).

4. Let \( T \) be a linear operator on a finite dimensional inner product space \( V \) over \( \mathbb{C} \). Let \( W \) be a \( T \)-invariant subspace of \( V \). Let \( W^\perp \) be the orthogonal complement of \( W \).

(a) (5) Show that \( W^\perp \) is \( T^* \)-invariant.

(b) (5) If \( T \) is normal and \( W \) is a span of some eigenvectors of \( T \), then \( W^\perp \) is both \( T \) and \( T^* \) invariant. (Note: \( T \) is normal if \( T^*T = TT^* \).)

(c) (10) If \( T \) is normal and \( W \) is \( T \)-invariant, show that \( W \) is also \( T^* \)-invariant.

(d) (5) Show that if \( T \) is normal and \( W \) is both \( T \) and \( T^* \) invariant, then \( T|_W \) is normal on \( W \).

5. (20) Let \( T \) be a linear operator on a finite dimensional inner product space \( V \) over \( \mathbb{C} \). Prove that \( T \) is self-adjoint if and only if \( \langle T\alpha | \alpha \rangle \) is real for every \( \alpha \) in \( V \). (Note: \( T \) is self-adjoint if \( T^* = T \).)
Problem 1. Let $A$ be an $n \times n$ matrix such that $\text{rk}(A) \leq 1$. Show that $\text{tr}(A) + 1 = \det(A + 1)$.

Problem 2. Count the largest number of pairwise nonsimilar $7 \times 7$ complex matrices $A$ such that $\text{rk}(A - 2) = 5$ and $f(x) = (x - 2)^3(x - 3)^2$ is
a) the minimal annihilating polynomial of $A$;
b) an annihilating polynomial of $A$.

Problem 3. Let $A, B, C$ be $n \times n$ matrices and let $A$ be invertible. True or false? Justify your answer!
a) $\text{rk}(AB) = \text{rk}(BA)$;
b) $\text{rk}(BC) = \text{rk}(CB)$;
c) $\text{rk}(ABC) = \text{rk}(BAC)$.

Problem 4. Let $A$ be a $3 \times 3$ matrix such that $\text{tr}(A) = \text{tr}(A^2) = \text{tr}(A^3) = 2$. Show that $A$ is not invertible.

Problem 5. What is the largest possible dimension of a subspace $W \subset \mathbb{R}^4$ such that the restriction of the form $Q(x_1, x_2, x_3, x_4) = x_1x_2 + x_2x_3 + x_3x_4$ to $W$ is positive definite?

Problem 6. Let $W_n$ be a subspace of the space $M(n, \mathbb{C})$ of complex $n \times n$ matrices spanned by matrices of the form $[A, B]$, i.e. $W_n = \{ AB - BA \mid A, B \in M(n, \mathbb{C}) \}$. Show $\dim W_n = n^2 - 1$. 
Throughout this examination, $m$ denotes the Lebesgue measure on $\mathbb{R}$.

**Problem 1:** Show that there is a compact set $K \subset [0, 1]$ with $m(K) \geq 11/12$, but such that $K$ contains no non-empty open intervals.

**Problem 2:** Suppose $(f_n)$ is a sequence in $L_1[0, 1]$ such that $\lim_{n \to \infty} \int_0^1 |f_n(x)| \, dm(x) = 0$. Show that for all $\delta > 0$, we have $m\{x \in [0, 1] : |f_n(x)| \geq \delta\} \to 0$ as $n \to \infty$.

**Problem 3:** Let $f \geq 0$ be a measurable function on $[0, 1]$. Assume $m\{f \geq x\} \leq 1/x^2$ for all $x \geq 1$. Show that $f \in L_r[0, 1]$ for all $1 \leq r < 2$.

**Problem 4:** Show that for $p, 1 < p \leq \infty$, $L_p[0, 1] \subset L_1[0, 1]$ but $L_p(\mathbb{R})$ is not a subset of $L_1(\mathbb{R})$.

**Problem 5:** Let $(r_n)$ be the sequence of functions such that $r_n : [0, 1) \to \{-1, 1\}$ given by the rule that $r_n(x) = (-1)^k$ for $x \in [\frac{k}{2^n}, \frac{k+1}{2^n})$ with $k = 0, \ldots, 2^n - 1$. Let $f \in L_1[0, 1]$. Show that $\lim_{n \to \infty} \int_0^1 f(x) r_n(x) \, dm(x) = 0$.

**Problem 6:** Show that for $f \in L_1(\mathbb{R})$, we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} |f(x + 1/n) - f(x)| \, dm(x) = 0.$$
1. (20) Let $A \in M_{m \times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. Prove the following.

(a) The equation $AX = b$ has a solution if and only if $b \in \ker(A^T)^\perp$.

(b) $A$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if $A^T X = 0$ has only trivial solution.
2. (20) Let $A = (a_{ij})$ be an $n \times n$ matrix with complex entries. Show that if

$$|a_{ii}| > \sum_{j=1, (j \neq i)}^{n} |a_{ij}|$$

for $i = 1, 2, \ldots, n$, then $A$ is invertible.
3. (20) Let $T : V \rightarrow V$ be a linear transformation on a finite dimensional vector space $V$ over a field $F$. Show that the following are equivalent.

(a) $T$ has a cyclic vector;

(b) If $TS = ST$ for some operator $S$ on $V$, then $S = g(T)$ for some polynomial $g$ over $F$. 
4. (20) Let $T : V \rightarrow V$ be a linear transformation on a finite dimensional vector space $V$ over a field $F$. Show that the following are equivalent.

   (a) Every $T$-invariant subspace of $V$ has a $T$-invariant complement.

   (b) The minimal polynomial $p_T$ of $T$ is a product of distinct monic and irreducible polynomials in $F[x]$:
5. (20) Let $A = (a_{ij})$ be an $n \times n$ matrix with complex entries. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of $A$ counting multiplicity. Show that

$$
\sum_{i,j} |a_{ij}|^2 \geq \sum_{i=1}^{n} |\lambda_i|^2.
$$

Moreover, equality holds if and only if $A$ is normal. (Hint: $\sum_{i,j} |a_{ij}|^2 = \text{Tr}(A^*A)$.)
Problem 1. Let $V$ be a vector space. Let $\{v_1, v_2, \ldots, v_k\}$ and $\{w_1, w_2, \ldots, w_l\}$ be two linearly independent sets of vectors in $V$. Assume $l > k$. Show that there exists $i \in \{1, \ldots, l\}$ such that the set $\{v_1, v_2, \ldots, v_k, w_i\}$ is linearly independent.

Problem 2. Let $V_n = \text{Mat}_R(n,n)$ be the space of real matrices of size $n \times n$. Let $A_n, U_n, D_n \subset V$ be the subspaces of skew-symmetric, strictly upper triangular matrices, and diagonal matrices respectively.

a) Show that $V_n = U_n \oplus A_n \oplus D_n$ is a direct sum decomposition.

b) Find the projection of the matrix $M$ to $U_3$ along $A_3 \oplus D_n$.

$$M = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 0 & 2 \\ 3 & -1 & 2 \end{pmatrix} \in V_3.$$ 

Problem 3. We say that $n \times n$ complex matrices $A$ and $B$ are equivalent if there exist invertible $n \times n$ complex matrices $X, Y$ such that $XAY = B$. In such a case we write $A \sim B$.

a) Show that $\sim$ is an equivalence relation.

b) True or false? If $A \sim B$ then $A^{-1} \sim B^{-1}$. Explain!

c) True or false? If $A \sim B$ then $A^2 \sim B^2$. Explain!

Problem 4. Consider the $n \times n$ matrix $A = (\delta_{i,n+1-i}a_i)$, where $a_i \in \mathbb{C}$:

$$A = \begin{pmatrix} 0 & 0 & \ldots & a_1 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & a_{n-1} & \ldots & 0 \\ a_n & 0 & \ldots & 0 \end{pmatrix}.$$ 

Give the minimal and characteristic polynomials of $A$. Give a sufficient and necessary condition for $A$ to be diagonalizable. Describe the Jordan canonical form of $A$.

Problem 5. Let $C : \mathbb{C}^4 \to \mathbb{C}^4$ be a linear operator given in basis $\{e_1, e_2, e_3, e_4\}$ by the matrix

$$C = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$ 

Describe all possible bases (in terms of $e_1, e_2, e_3, e_4$) in which $C$ is given by a matrix in a Jordan canonical form.

Problem 6. Let $B(x, y)$ be a bilinear form on $\mathbb{R}^{n+k}$ given by the formula

$$B(x, y) = \sum_{i=1}^{n} x_i y_i - \sum_{j=1}^{k} x_{n+j} y_{n+j}, \quad x, y \in \mathbb{R}^{n+k}.$$ 

What is the maximal dimension of the subspace $W \subset \mathbb{R}^{n+k}$ such that $B(x, y) = 0$ for all $x, y \in W$? Give an example of such a subspace.
1. (20) Let $A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}$.

(a) Find the invariant factors and minimal polynomials of $A$;
(b) Find the Jordan canonical form $J$ of $A$ and an invertible matrix $P$ such that $P^{-1}AP = J$.

(Note: $A$ has only one eigenvalue of multiplicity 4.)
2. (20) Let $A$ and $B$ be two $n \times n$ matrices over $\mathbb{Q}$ with the same minimal polynomial and the same characteristic polynomial.

(a) Prove that if $n = 3$, then $A$ and $B$ are similar.

(b) **Prove or disprove** that if $n = 4$, then $A$ and $B$ are similar.
3. (20) It is known that for an \( n \times n \) matrix \( A \) over \( \mathbb{R} \) the classical adjoint \( \text{adj} A \) of \( A \) satisfying:

\[
\text{rank}(\text{adj} A) = \begin{cases} 
  n & \text{if rank } A = n \\
  1 & \text{if rank } A = n - 1 \\
  0 & \text{if rank } A < n - 1.
\end{cases}
\]

For \( n \geq 2 \), show that

(a) \( \det(\text{adj} A) = (\det A)^{n-1} \);

(b) \( \text{adj}(\text{adj} A) = (\det A)^{n-2} A \).
4. (20) Let \( W \) be an invariant subspace of a matrix \( A \in M_{n \times n}(\mathbb{R}) \). Let \( f_A \) be the characteristic polynomial of \( A \). Prove the following.

(a) \( W^\perp \) is invariant under \( A^T \), where \( A^T \) is the transpose matrix of \( A \).

(b) \( f_A(x) = f_{A|W}(x)f_{A^T|W^\perp}(x) \), where \( A|W \) is the restriction of the linear transformation \( A \) to \( W \).
5. (20) Recall that an $n \times n$ matrix $A$ over $\mathbb{C}$ is normal if $A^*A = AA^*$, where $A^* = \bar{A}^T$. Show that if $A \in M_{3 \times 3}(\mathbb{R})$ is normal, then there is an orthogonal matrix $O$ such that $O^TAO$ is either diagonal or is in the form

$$
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & a & b \\
0 & -b & a
\end{pmatrix},
$$

where $b \neq 0$. 

1. (20) Let

\[ A = \begin{pmatrix} -7 & -1 & -1 \\ -21 & -3 & -3 \\ 70 & 10 & 10 \end{pmatrix}. \]

Find the Jordan canonical form \( J \) of \( A \) and an invertible matrix \( P \) such that \( P^{-1}AP = J \).
2. (20) Let $T$ be a linear operator on an $n$-dimensional vector space $V$ ($n > 1$) and $W$ be a $k$-dimensional ($0 < k < n$) $T$-invariant subspace. Show that if $T$ has $n$ distinct eigenvalues, then for any $T$-invariant direct sum decomposition of $V = W_1 \oplus W_2 \oplus \cdots \oplus W_s$, $W = (W_1 \cap W) \oplus (W_2 \cap W) \oplus \cdots \oplus (W_s \cap W)$. 
3. (20) Let $T$ be a linear transformation on a finite dimensional vector space over the field $\mathbb{F}$. Let $p_T$ and $f_T$ be the minimal and respectively characteristic polynomial of $T$. If $p_T = f_T = q^k$ for some irreducible polynomial $q \in \mathbb{F}[x]$ and $k > 1$, show that no nonzero proper $T$-invariant subspace can have a $T$-invariant complement.
4. (20) Let \( A \) be the \( 2n \times 2n \) complex matrix

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & a_{2n} \\
0 & 0 & \cdots & a_{2n-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a_2 & \cdots & 0 & \cdots \\
a_1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

Find a necessary and sufficient condition that the matrix \( A \) is diagonalizable. Justify your answer.
5. (20) Recall that an operator \( N \) on a finite dimensional inner product space is normal if \( N^*N = NN^* \). Show that the product \( ST \) of two normal operators \( S \) and \( T \) on a finite dimensional inner product space \( V \) is normal if \( ST = TS \).
Math 554 Qualifying Exam
January, 2013

Make sure to provide detailed arguments to support your claims! For each problem you may prove one part by using the claims of the other parts.

1. Let \( A = \begin{pmatrix} -5 & 1 & 1 \\ -3 & -1 & 1 \\ -6 & 2 & 0 \end{pmatrix} \)
   a) (10) Find the characteristic polynomial and minimal polynomial of \( A \).
   b) (5) Determine if \( A \) is diagonalizable or not.
   c) (5) Find the rational form of \( A \).
   d) (5) Find the Jordan canonical form \( J \) of \( A \).
   e) (10) Find an invertible matrix \( P \) such that \( P^{-1}AP = J \).
   (Make sure to double check part a)!

2. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the linear transformation defined by reflecting a vector in \( \mathbb{R}^2 \) with respect to the line \( y - \sqrt{3}x = 0 \) followed by reflecting the resulting vector with respect to the line \( y + x = 0 \).
   a) (10) Find a matrix representation of \( T \) with respect to the standard basis \( \mathcal{E} = \{(1, 0), (0, 1)\} \).
   b) (5) Show that \( T \) is a (counter-clockwise) rotation and find the angle of the rotation.

3. Let \( T \) be a linear operator on an \( n \)-dimensional vector space \( V \). Let \( R(T) \) be the range of the operator \( T \) on \( V \) and \( N(T) \) be the null space of \( T \).
   a) (7) Let \( R_\infty(T) = \cap_{k=1}^\infty R(T^k) \). Show that \( R_\infty(T) = R(T^n) \).
   b) (8) Let \( N_\infty(T) = \cup_{k=1}^\infty N(T^k) \). Show that \( N_\infty(T) = N(T^n) \).

4. (15) Let \( S \) and \( T \) be two commuting linear operators on a finite dimensional vector space \( V \). If the minimal and characteristic polynomials of \( T \) are equal, show that \( S \) is a polynomial in \( T \).

5. Let \( A \) be an \( n \times n \) antisymmetric matrix: \( A^T = -A \) over \( \mathbb{R} \).
   a) (10) Show that there is an invertible \( n \times n \) matrix \( P \) such that \( PAP^T \) is equal to the block diagonal matrix \( J = \text{diag}\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \} \), where the last 0 in the matrix is a square 0 matrix of certain dimension. (Hint: use both elementary row and column operations.)
   b) (10) Show that there are symmetric matrices \( C \) and \( D \) such that \( A = CD - DC \).
1. (10) Show that if \( f \) is a polynomial in \( \mathbb{C}[x] \) of degree \( n \geq 2 \), then \( f' | f \) if and only if \( f(x) = a(x - c)^n \) for some \( c \in \mathbb{C} \).
2. (15) Let $\text{adj}A$ be the classical adjoint of an $n \times n$ matrix $A$ over $\mathbb{R}$. Show that

$$\text{rank}(\text{adj}A) = \begin{cases} n & \text{if } \text{rank}A = n \\ 1 & \text{if } \text{rank}A = n - 1 \\ 0 & \text{if } \text{rank}A < n - 1. \end{cases}$$
3. (15) Let \( A \) be the \( n \times n \) matrix

\[
\begin{pmatrix}
  x & a & a & a & \cdots & a \\
  -a & x & a & a & \cdots & a \\
  -a & -a & x & a & \cdots & a \\
  -a & -a & -a & x & \cdots & a \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  -a & -a & -a & -a & \cdots & x
\end{pmatrix}
\]

Find the determinant of \( A \).
4. (10) If $A$ is an invertible matrix over $\mathbb{R}$, show that there are positive constants $c_1 < c_2$ such that

$$c_1 X^T X \leq X^T A^T A X \leq c_2 X^T X,$$

for all $X \in \mathbb{R}^{n \times 1}$. 
5. (10) Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be linearly independent in a vector space \( V \) over the field \( \mathbb{F} \). Assume that \( A \) is an \( n \times k \) matrix over \( \mathbb{F} \) and

\[
(\beta_1, \beta_2, \ldots, \beta_k) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \cdot A.
\]

Show that the dimension of the subspace \( W := \text{span}\{\beta_1, \beta_2, \ldots, \beta_k\} \) is the rank of \( A \).
6. (10) Find the minimal polynomial of the matrix

\[ A = \begin{pmatrix} 4 & 1 & 0 & -1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix}. \]
7. (15) Find the Jordan form $J$ of the matrix $A = \begin{pmatrix} 13 & 16 & 16 \\ -5 & -7 & -6 \\ -6 & -8 & -7 \end{pmatrix}$, and find an invertible matrix $P$ such that $P^{-1}AP = J$. 


1. (15) Let $A$ and $B$ be two $n \times n$ matrices over $\mathbb{R}$ which are similar over $\mathbb{C}$. Prove or disprove that $A$ is similar to $B$ over $\mathbb{R}$.
2. (15) Let $A$ be an $n \times n$ matrix with entries in $\mathbb{C}$ such that for each eigenvalue $c$ of $A$, the eigensubspace of $A$ associated with $c$ is one dimensional. Let $B$ be any $n \times n$ matrix satisfying $AB = BA$. Show that there is a polynomial $f \in \mathbb{C}[x]$ such that $B = f(A)$. 
3. (15) Let $V$ be a finite dimensional vector space over the field $F$ and $T$ be a linear transformation on $V$. For any vector $\alpha \in V$, let $Z(\alpha; T) = \{ f(T)\alpha | f \in \mathbb{F}[x] \}$. Suppose that $V$ has two cyclic decompositions:

$V = Z(\beta_1; T) \oplus Z(\beta_2; T)$, with the $T$-annihilator of $\beta_i$ being $q_i$ and satisfying $q_1$ divides $q_2$;

$V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T)$, with the $T$-annihilator of $\alpha_i$ being $p_i$ satisfying $p_i$ divides $p_{i+1}$ for $i = 1, 2, \ldots, r - 1$.

Show that $r = 2$, $p_1 = q_1$ and $p_2 = q_2$. 
4. (30) Let \( A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 35 & 21 & 1 & 0 \\ -15 & -9 & 0 & 1 \end{pmatrix} \).

a) Find the invariant factors, minimal polynomial and the characteristic polynomial of \( A \).

b) Find the Jordan canonical form \( J \) of \( A \) and an invertible matrix \( P \) such that \( P^{-1}AP = J \).
5. (15) Let $T$ be a linear operator on the $n$-dimensional inner product space $V$ over $\mathbb{C}$ satisfying the property that $TT^* = f(T)$, where $f$ is any polynomial in $\mathbb{C}[x]$ such that $f(0) = 0$. Show that $T$ is normal. (Recall that the adjoint operator $T^*$ is defined by the equality $(T\alpha|\beta) = (\alpha|T^*\beta)$ for all vectors $\alpha$ and $\beta$ in $V$ and $T$ is normal if $TT^* = T^*T$.)
1. (20) Prove that the determinant of the $n \times n$ matrix
\[
D_n = \begin{pmatrix}
a & b & b & \cdots & b \\
c & a & b & \cdots & b \\
c & c & a & \cdots & b \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c & c & c & \cdots & a
\end{pmatrix}
\]
is
\[
\frac{c(a-b)^n - b(a-c)^n}{c-b}
\]
for $b \neq c$.

2. (20) Let $E_1, E_2, \ldots, E_n$ be projections on a vector space $V$ over a field $F$ of characteristic 0, such that, for each $k$ with $1 \leq k \leq n$, $E_1 + E_2 + \cdots + E_k$ is also a projection. Prove that $E_i E_j = E_j E_i = 0$ for all $1 \leq i < j \leq n$.

3. (20) Let $A$ be an invertible $n \times n$ matrix over the field of complex numbers. Show that
a) $A$ and $A^{-1}$ have the same eigensubspaces.

b) If $J = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & \cdots \\ & & & J_k \end{pmatrix}$, where $J_i = \begin{pmatrix} c_i & 1 \\ c_i & 1 \\ & & \ddots & 1 \\ & & c_i & 1 \end{pmatrix}_{k_i \times k_i}$, is the Jordan form for $A$, then the Jordan form for $A^{-1}$ is $J' = \begin{pmatrix} J'_1 & & \\ & J'_2 & \\ & & \cdots \\ & & & J'_k \end{pmatrix}$, where
\[
J'_i = \begin{pmatrix} c_i^{-1} & 1 \\ c_i^{-1} & 1 \\ & & \ddots & 1 \\ & & c_i^{-1} & 1 \end{pmatrix}_{k_i \times k_i}
\]
for each $i$. 
4. (20) Recall that a linear transformation $T$ on a finite dimensional inner product space $V$ over $\mathbb{C}$ is normal if and only if $T^*T = TT^*$. Show that the following statements are equivalent for a linear transformation $T$ on $V$.

(a) $T$ is normal on $V$;
(b) $\|T\alpha\| = \|T^*\alpha\|$ for every vector $\alpha \in V$;
(c) $T^*$ is a polynomial (with complex coefficients) in $T$.

5. (20) Let $T$ be a linear transformation on the finite-dimensional vector space $V$, let $p = p_1^{r_1} \cdots p_k^{r_k}$ be the minimal polynomial for $T$ (where $r_i > 0$), and let $V = W_1 \oplus \cdots \oplus W_k$ be the primary decomposition for $T$, i.e. $W_j$ is the null space of $p_j(T)^{r_j}$. Let $W$ be any $T$-invariant subspace of $V$. Show that $W = (W \cap W_1) \oplus (W \cap W_2) \oplus \cdots \oplus (W \cap W_k)$. 
1. (20) Let $W_1$ and $W_2$ be subspaces of a finite dimensional inner product space $V$. Show that
(a) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$;
(b) $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$, where $W^\perp$ denotes the orthogonal complement of the subspace $W$ in $V$. 

Make sure to double check your solutions!
2. (20) Let \( A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 35 & 21 & 1 & 0 \\ -15 & -9 & 0 & 1 \end{pmatrix} \).

a) Find the invariant factors, minimal polynomial and the characteristic polynomial of \( A \).

b) Find the Jordan canonical form \( J \) of \( A \) and an invertible matrix \( P \) such that \( P^{-1}AP = J \).
3. (20) Let $T : V \rightarrow V$ be a linear transformation on the finite dimensional vector space $V$ over a field $F$. If $T^m = T$ for some positive integer $m > 1$, show that

(a) $\text{Null}(T) \cap \text{Im}(T) = \{0\}$, where $\text{Null}(T)$ and $\text{Im}(T)$ are respectively the null space of $T$ and the image of $T$;

(b) $\text{Null}(T) = \text{Null}(T^k)$ for any positive integer $k$.  

4. (20) Let $T : V \rightarrow V$ be a linear transformation on the finite dimensional vector space $V$ over $\mathbb{C}$. Suppose that $T^3 + 3T = I$, show that $T$ is diagonalizable.
5. (20) Let $T : V \rightarrow V$ be a linear transformation on the finite dimensional vector space $V$ over a field $F$. Let $W$ be a nonzero proper $T$-invariant subspace of $V$. Suppose that the characteristic polynomial $f_T$ of $T$ satisfies that $f_T(0) \neq 0$. Show that if $T$ has a cyclic vector, then $T|_W : W \rightarrow W$ has a cyclic vector.
Math 554 Qualifying Exam

August, 2010

Ron Ji

You may use any theorems from the textbook. Any other claims must be proved in details.

1. (20) Let $V$ be the $\mathbb{R}$-vector space consisting of real polynomials of degree not exceeding $n$. Let $(t_0, t_1, ..., t_n)$ and $(t'_0, t'_1, t'_2, ..., t'_n)$ be $(n + 1)$-tuples of distinct real numbers. Define $L_i(f) = f(t_i)$ and $L'_i(f) = f(t'_i)$ for $f \in V$ and $i = 0, 1, 2, ..., n$.

(a) Show that both sets $\{L_0, L_1, ..., L_n\}$ and $\{L'_0, L'_1, ..., L'_n\}$ are bases for the dual space $V^\ast$.

(b) Find an $n \times n$ invertible real matrix $P = (P_{ij})$ (in terms of $t_i$’s and $t'_j$’s) such that $L'_j = \sum_{i=0}^{n} P_{ij} L_i$. 

$p_{11} = 1$
2. (30) Let \( A = \begin{pmatrix} 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 3 \end{pmatrix} \).

(a) Find the invariant factors, minimal and characteristic polynomials of \( A \).

(b) Find the Jordan canonical form \( J \) of \( A \) and an invertible matrix \( P \) such that \( P^{-1}AP = J \).
3. (15) Let $T$ be a linear transformation on a finite dimensional vector space over the field $\mathbb{F}$. If the minimal polynomial of $T$ is irreducible, prove the following.

(a) Every $T$-invariant subspace $W$ of $V$ is $T$-admissible, that is, for any polynomial $f \in \mathbb{F}[x]$ if $f(T)\alpha \in W$, then there is $\beta \in W$ such that $f(T)\alpha = f(T)\beta$.

(b) Every $T$-invariant subspace $W$ has a $T$-invariant complement $W'$ satisfying: (i) $W + W' = V$, and (ii) $W \cap W' = \{0\}$. 
4. (15) (a) Suppose that $A$ and $B$ are $3 \times 3$ matrices over the field $F$. Show that $A$ is similar to $B$ if and only if $A$ and $B$ have the same minimal polynomial and the same characteristic polynomial.

(b) Is the statement in (a) true or not in general?
5. (10) Let \( \mathbb{F} \) be a field. Suppose that \( A \in M_{m \times n}(\mathbb{F}) \) and \( B \in M_{n \times m}(\mathbb{F}) \). Show that

(a) \( \det(\mathbf{I}_m - AB) = \det(\mathbf{I}_n - BA) \).

(b) if \( n \leq m \), then \( \det(x\mathbf{I}_m - AB) = x^{m-n}\det(x\mathbf{I}_n - BA) \).
6. (20) If each $A_i$ is symmetric $n \times n$ real matrix and $\sum_{i=1}^{m} A_i = I_n$, where $I_n$ is the $n \times n$ identity matrix, then the following conditions are equivalent:

1. $A_i^2 = A_i$, for $i = 1, 2, 3, ..., m$;
2. $A_i A_j = 0$, whenever $1 \leq i, j \leq m$ and $i \neq j$;
3. $\sum_{i=1}^{m} \text{rank} A_i = n$.

(Hint: For certain directions the trace function or induction might help.)
1. (20) Let $V$ be the vector space of $n \times n$ matrices over the field $\mathbb{F}$ and $V_0$ be the subspace consisting of matrices of the form $C = AB - BA$ for some $A$, $B$ in $V$. Prove that $V_0 = \{A \in V | \text{Trace}(A) = 0\}$.

2. (15) If $W$ is a subspace of a finite dimensional vector space $V$ and if $\{g_1, g_2, \ldots, g_r\}$ is a basis of $W^0 = \{f \in V^* | f|_W = 0\}$, then $W = \bigcap_{i=1}^r N_{g_i}$, where for $f \in V^*$, $N_f = \{ \alpha \in V | f(\alpha) = 0\}$.

3. (25) Let $\langle A|B \rangle = \text{Trace}(AB^*)$ be the inner product on $M_n(\mathbb{C})$, the $n \times n$ matrices over the field $\mathbb{C}$. Let $A$ be in $M_n(\mathbb{C})$ and $T_A : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be the linear transformation defined by $T_A(B) = ABA^*$, where $A^* = \bar{A}^t$. Show that
   (1) $T_A$ is invertible if and only if $A$ is invertible.
   (2) $T_A$ is unitary if and only if $A$ is unitary.
   (3) $T_A$ is self-adjoint if and only if $A$ is normal and $\lambda \bar{\mu}$ is real for any two eigenvalues $\lambda$ and $\mu$ of $A$.
   (Hint: If $T_A$ is self-adjoint, show that $A$ is normal first; and if $A$ is normal, there is a unitary matrix $U$ and a diagonal matrix $D$ such that $UDU^* = A$.)

4. (20) Let $T$ be a linear transformation on the finite dimensional vector space $V$ over the field $\mathbb{F}$. Let $p$ be the minimal polynomial of $T$. If $p = g_1g_2$, where $g_1$ and $g_2$ are relatively prime factors of $p$, show that
   (a) $V = W_1 \oplus W_2$, where $W_i = \{ \alpha \in V | g_i(T) \alpha = 0\}$, for $i = 1, 2$.
   (b) $W_1$ and $W_2$ are $T$-invariant.
   (c) If $T_i$ is the operator on $W_i$ induced by $T$, then the minimal polynomial for $T_i$ is $g_i$ for $i = 1, 2$.

5. (20) Let $A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.
   a) Find the minimal polynomial and the characteristic polynomial of $A$.
   b) Find the Jordan canonical form $J$ of $A$ and an invertible matrix $P$ such that $P^{-1}AP = J$. 

You may use any theorems from the textbook. Any other claims must be proved in details.
1. (10) Let $V$ be an $n$-dimensional vector space over a field $F$. Show that the vectors $\alpha_1, \alpha_2, ..., \alpha_n$ form a basis for $V$ if and only if for any nonzero linear functional $f$ on $V$, there is a nonzero vector $\alpha$ in the span of $\alpha_1, \alpha_2, ..., \alpha_n$ such that $f(\alpha) \neq 0$.

2. (15) Let $W_1$ and $W_2$ be subspaces of a finite dimensional inner product space $V$. Show that $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$, where $W^\perp$ denotes the orthogonal complement of the subspace $W$ in $V$.

3. Let $A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}$.

   a) (10) Find the minimal polynomial and the characteristic polynomial of $A$.

   b) (15) Find the Jordan canonical form $J$ of $A$ and an invertible matrix $P$ such that $P^{-1}AP = J$.

4. (20) Suppose that $A$ and $B$ are $n \times n$ matrices such that $AB = 0$. Show that $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B) \leq n$.

5. (20) Let $T$ be a linear transformation on a finite dimensional vector space $V$ over a field $F$. Let $c$ be a characteristic value of $T$ and $W_c$ be the characteristic subspace of $T$ associated with $c$. Suppose that a proper $T$-invariant subspace $W$ contains $W_c$, and there is a vector $\alpha$ in $V$ but not in $W$ such that $(T - cI)\alpha$ is in $W$. Show that the minimal polynomial $p_T$ of $T$ is in the form $(x - c)^2q$ for some nonzero polynomial $q$ in $F[x]$.

6. (20) Let $T$ be a linear transformation on a finite dimensional vector space $V$ over a field $F$. If the minimal polynomial $p_T$ of $T$ is irreducible, then every $T$-invariant subspace $W$ has a $T$-invariant complement $W'$. That is, $TW \subset W$, $TW' \subset W'$, $W + W' = V$ and $W \cap W' = \{0\}$. 

Any points over 100 will be bonus. You may use any theorems stated in the textbook or proved in the lectures.