IUPUI Qualifying Exam in Real Analysis

Summer 2018

Pavel Bleher

Problem 1. Let E be the set of real numbers x on the interval $[0,1]$ such that in the decimal form of $x = 0.i_1 i_2 i_3 \ldots$ there is no string of four consecutive digits 2018. Prove that

(1) the set E is uncountable;
(2) mE=0, where mE is the Lebesgue measure of E.

Problem 2. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{|x - n^{-1}|^{1/2}}{n^2}$$

is absolutely continuous on $[0,1]$.

Problem 3. Prove that for any integrable function f on the interval $[a,b]$,

$$\lim_{n\to\infty} \int_a^b f(x)\, \sin^4(nx)\, dx = \frac{3}{8} \int_a^b f(x)\, dx.$$

Problem 4. Let $f(x) \in L^2(-\infty,\infty)$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) f(x+n) dx = 0.$$

Problem 5. Let $f \in L^3[0,\pi]$ and

$$g(x) = \frac{f(x)}{|\sin x|^{0.1}}\,.$$

Prove that $\|g\|_2 \le 2\|f\|_3$.

Problem 6. Let $f(x)$ be an integrable function on $(-\infty,\infty)$ and $n \ge 1$ an integer. Define

$$f_n(x) = n \int_x^{x+\frac{1}{n}} f(t)\, dt\,.$$

Prove that $\|f_n\|_1 \le \|f\|_1$.

IUPUI Qualifying Exam in Real Analysis

Winter 2018

Pavel Bleher

Problem 1. A real number x on [0,1] is called a *Liouville type number*, if its decimal representation has the form

$$x = \sum_{j=1}^{\infty} \frac{1}{10^{n_j}},$$

where $n_j \in \mathbb{N}$ are such that

$$n_{j+1} - n_j \geq j!, \quad j = 1, 2, \dots .$$

(Liouville proved that all such numbers are transcendental.) Prove that the set of Liouville type numbers is uncountable.

Problem 2. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{|\sin(nx)|}}$$

converges almost everywhere on $[0, 2\pi]$.

Problem 3. Let $f(x)$ be an absolutely continuous function on an interval $[a, b]$ such that $f'(x) \in L^3[a, b]$. Prove that there exists a constant $C > 0$ such that

$$|f(y) - f(x)| \leq C|y - x|^{2/3}$$

for all $a \leq x < y \leq b$.

Problem 4. Let $f(x) \in L^1(-\infty, \infty)$. Prove that

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} \frac{f(x)\,dx}{1 + (x - a)^2} = 0.$$

Problem 5. Let $f \in L^6[0, 1]$, $g \in L^7[0, 1]$, and $h \in L^8[0, 1]$. Prove that $fgh \in L^2[0, 1]$.

Problem 6. Let $E_1, E_2, \dots$ be a sequence of measurable sets on the line such that $m(E_j \triangle E_k) < \infty$ for all $j, k = 1, 2, \dots$ and

$$\lim_{j,k\to\infty} m(E_j \triangle E_k) = 0, \quad A \triangle B := (A \setminus B) \cup (B \setminus A).$$

Prove that there exists a measurable set E such that

$$\lim_{j\to\infty} m(E_j \triangle E) = 0.$$

IUPUI Qualifying Exam in Real Analysis

Summer 2017

Pavel Bleher

Problem 1. A real number is called transcendental if it is not a root of a polynomial with integer coefficients. Let E be the set of transcendental numbers on the interval $[0,1]$. Prove that the set E is measurable and $mE = 1$.

Problem 2. Let A and E be disjoint sets on the line and let E be measurable. Prove that

$$m^*(A \cup E) = m^*A + mE.$$

Problem 3. Prove that for any integrable function f on the interval $[a,b]$,

$$\lim_{n\to\infty} \int_a^b f(x)\,|\sin(nx)|\,dx = \frac{2}{\pi}\int_a^b f(x)\,dx.$$

Problem 4. Let $f(x) \in L^1(-\infty,\infty)$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x-n)\arctan\left(\frac{x}{n}\right) dx = \frac{\pi}{4}\int_{-\infty}^{\infty} f(x)\,dx.$$

Problem 5. Let $f(x)$ be a nonnegative integrable function on the interval $[a,b]$. Prove that

$$\left(\int_a^b f(x)\sin x\,dx\right)^2 + \left(\int_a^b f(x)\cos x\,dx\right)^2 \le \left(\int_a^b f(x)\,dx\right)^2.$$

Problem 6. Let $f(x) \in L^2[0,1]$. Define

$$g(x) = \frac{1}{x^{4/3}}\int_0^x f(t)\,dt\,.$$

Prove that $\|g\|_1 \le 6\|f\|_2$.

IUPUI Qualifying Exam in Real Analysis
Winter 2017

Pavel Bleher

Problem 1. Let A and B be two sets on the interval $[0, 1]$ such that

$$\overline{A} \cap \overline{B} = \emptyset,$$

where $\overline{A}$ and $\overline{B}$ are the closures of the sets A and B, respectively, and $\emptyset$ is the empty set. Prove that

$$m^*\big(A \cup B\big) = m^*A + m^*B,$$

where m^* is the outer measure.

Problem 2. Let $f(x)$ be an absolutely continuous function on $[0, 1]$ and E a set on [0,1] such that $mE = 0$, where mE is the Lebesgue measure of the set E. Let $f(E)$ be the image of the set E with respect to f,

$$f(E) = \{y \in \mathbb{R} \,|\, \exists\, x \in [0, 1] \text{ such that } f(x) = y\}.$$

Prove that $mf(E) = 0$.

Problem 3. Consider the set E of numbers x on $[0, 1]$ such that in every 100 consecutive digits $i_{k+1}, \ldots, i_{k+100}$, $k = 0, 1, 2, \ldots$, of a decimal fraction representing x, $x = 0.i_1 i_2 i_3 \ldots$, where $i_k \in \{0, 1, \ldots, 9\}$, there is at least one 0. Prove that $mE = 0$.

Problem 4. Let $f(x)$ be an integrable function on the interval $[a, b]$. Define recursively the functions,

$$f_0(x) = f(x), \quad f_k(x) = \int_a^x f_{k-1}(t)\,dt, \quad k \ge 1; \quad a \le x \le b.$$

Prove that the series

$$F(x) = \sum_{k=1}^{\infty} f_k(x)$$

converges uniformly on $[a, b]$ and $F(x)$ is an absolutely continuous function.

Problem 5. Let A be a measurable subset of $[0, 2\pi]$. Prove that

$$\lim_{n\to\infty} \int_A \cos^2(nx)\,dx = \frac{1}{2}\,mA,$$

where mA is the measure of A.

Problem 6. Let $f(x)$ be an integrable function on $[0, \infty)$. Prove that

$$\lim_{n\to\infty} n \int_0^1 x f(nx)\,dx = 0.$$

Qualifying Exam in Real Analysis
IUPUI, January 2016
Joseph Rosenblatt

Problem 1: Show that there is a compact set $K \subset [0,1]$ with Lebesgue measure $m(K) > 0$, but such that K contains no non-empty open interval.

Problem 2: Suppose (E_n) is a sequence of measurable subsets of $[0,1]$ with Lebesgue measure $m(E_n \Delta E_{n+1}) \leq 1/2^n$ for all $n \geq 1$. Here $A\Delta B = (A\backslash B) \cup (B \backslash A)$.

a) Show that (1_{E_n}) is a Cauchy sequence in L_1-norm in $L_1[0,1]$.

b) Show that there is a measurable set E such that $m(E\Delta E_n) \to 0$ as $n \to \infty$.

Problem 3: Let $f \geq 0$ be a measurable function on $[0,1]$. Show that if $\sum_{n=1}^{\infty} m\{f \geq n^2\} < \infty$, then $\int_0^1 \sqrt{f(x)}\, dm(x) < \infty$.

Problem 4: Describe a measurable function f on $[0,1]$ such that $f \in L_1[0,1]$, but for all $p, 1 < p \leq \infty$, $f \notin L_p[0,1]$.

Problem 5: Suppose $h : [0,1] \to \mathbb{R}$ is continuous and $\int_0^1 h(x)\, dx = 0$. For $x \geq 0$, let $\{x\}$ be the fractional part of x: $\{x\} = x - p$ where p is a positive integer such that $p \leq x < p+1$.

a) Show that for any $0 \leq a \leq b \leq 1$, we have $\int_a^b h(\{nx\})\, dx \to 0$ as $n \to \infty$.

b) Using a), show that for any $g \in L_1[0,1]$, we have $\int_0^1 h(\{nx\})\, g(x)\, dx \to 0$ as $n \to \infty$.

Problem 6: Show that for $f \in L_1(\mathbb{R})$ and any $\epsilon > 0$, we have

$$\lim_{n\to\infty} m\{x \in \mathbb{R} : |f(x + \frac{1}{n}) - f(x)| \geq \epsilon\} = 0.$$

Qualifying Exam in Real Analysis
IUPUI, August 2016
Joseph Rosenblatt

Throughout this examination, m denotes the Lebesgue measure on $\mathbb{R}$.

Problem 1: Show that there is a compact set $K \subset [0,1]$ with $m(K) \geq 11/12$, but such that K contains no non-empty open intervals.

Problem 2: Suppose (f_n) is a sequence in $L_1[0,1]$ such that $\lim\limits_{n\to\infty} \int_0^1 |f_n(x)|\,dm(x) = 0$. Show that for all $\delta > 0$, we have $m\{x \in [0,1] : |f_n(x)| \geq \delta\} \to 0$ as $n \to \infty$.

Problem 3: Let $f \geq 0$ be a measurable function on $[0,1]$. Assume $m\{f \geq x\} \leq 1/x^2$ for all $x \geq 1$. Show that $f \in L_r[0,1]$ for all $1 \leq r < 2$.

Problem 4: Show that for $p, 1 < p \leq \infty$, $L_p[0,1] \subset L_1[0,1]$ but $L_p(\mathbb{R})$ is not a subset of $L_1(\mathbb{R})$.

Problem 5: Let (r_n) be the sequence of functions such that $r_n : [0,1) \to \{-1,1\}$ given by the rule that $r_n(x) = (-1)^k$ for $x \in [\frac{k}{2^n}, \frac{k+1}{2^n})$ with $k = 0, \ldots, 2^n - 1$. Let $f \in L_1[0,1]$. Show that $\lim\limits_{n\to\infty} \int_0^1 f(x) r_n(x)\,dm(x) = 0$.

Problem 6: Show that for $f \in L_1(\mathbb{R})$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x + 1/n) - f(x)|\,dm(x) = 0.$$

IUPUI Qualifying Exam in Real Analysis, Winter 2015

Pavel Bleher

Problem 1. Let $E \subset [0,1]$ be a measurable set such that there exists $\varepsilon > 0$ such that

$$m(E \cap [a,b]) \geq \varepsilon\, |b-a|$$

for all $[a,b] \subset [0,1]$. Prove that $mE = 1$.

Problem 2. Let E be a measurable set on the line and $f(x)$ a nonnegative measurable function on E such that the limit,

$$L = \lim_{n\to\infty} \int_E [f(x)]^n dx,$$

exists, where $0 < L < \infty$. Prove that $m\{x \in E \mid f(x) = 1\} = L$.

Problem 3. Let f be an integrable function on a set E of a finite Lebesgue measure $mE < \infty$. Let $E_n = \{x \in E \mid |f(x)| \geq n\}$. Prove that

$$\sum_{n=0}^{\infty} mE_n \geq \int_E |f(x)|\, dx \geq \sum_{n=1}^{\infty} mE_n.$$

Problem 4. Prove that if f is absolutely continuous on $[a,b]$ and $g(x) = e^{|f(x)|}$ then g is absolutely continuous on $[a,b]$ as well.

Problem 5. Prove that if $f \in L^1[0,\infty)$, then

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n x f(x)\, dx = 0.$$

Problem 6. Prove that if $f(x) \in L^{4+\varepsilon}[0,1]$, where $\varepsilon > 0$, and $g(x) = f(x^2)$, then $g(x) \in L^2[0,1]$.

QUALIFYING EXAM IN REAL ANALYSIS

IUPUI, AUGUST 2015

PAVEL BLEHER

Problem 1. Let f be a measurable function on $[0,1]$ such that f is finite almost everywhere. Prove that for any $\varepsilon > 0$, there is a polynomial $P(x) = \sum_{k=0}^{n} p_k x^k$, such that

$$m\left\{x \,\middle|\, |f(x) - P(x)| \geq \varepsilon\right\} \leq \varepsilon,$$

where m is the Lebesgue measure.

Problem 2. Let $\mathcal{M}$ be the σ-algebra of all Lebesgue measurable sets on $[0,1]$. Prove that there is a countable collection of sets $\mathcal{D} \subset \mathcal{M}$ such that for every set $E \in \mathcal{M}$ and every $\varepsilon > 0$ there is a set $A \in \mathcal{D}$ such that

$$m(A\Delta E) \leq \varepsilon, \qquad A\Delta E \equiv (A \setminus E) \cup (E \setminus A).$$

Problem 3. Let f be a measurable function on $[0,1]$ such that $0 \leq f(x) \leq 1$ for all x on $[0,1]$, and let g be a function of bounded variation on $[0,1]$. Prove that the function $g(f(x))$ is measurable on $[0,1]$.

Problem 4. Let f be a continuous function on $[0,1]$, which is differentiable almost everywhere. Suppose that $f' \in L^p[0,1]$, where $p > 1$. Let $\alpha = 1 - \frac{1}{p}$. Prove that there is a constant $C > 0$ such that

$$|f(y) - f(x)| \leq C|y - x|^{\alpha},$$

for all x, y on $[0,1]$.

Problem 5. Let $f \in L^{\infty}[0,1]$. Prove that for every $p \geq 1$,

$$\exp\left[\int_0^1 f(x)\,dx\right] \leq \|e^{f(x)}\|_p.$$

Problem 6. Prove that for every $p < 1$ the function

$$f(x) = \left|\sin\left(x^{-1}\right)\right|^{-p}$$

is integrable on $[0,1]$.

IUPUI Qualifying Exam in Real Analysis, Winter 2014

Pavel Bleher

Problem 1. Let A, B and C be measurable sets on the line such that

$$\begin{aligned} &m(A \cap B) = 0.3, \quad m(A \cap C) = 0.4, \quad m(B \cap C) = 0.2, \\ &m(A \cup B) = 0.8, \quad m(A \cup C) = 0.8, \quad m(B \cup C) = 0.9, \\ &m(A \cup B \cup C) = 1. \end{aligned}$$

where m is the Lebesgue measure. Find $m(A \cap B \cap C)$.

Problem 2. For real numbers a, b, c, let $s[a, b, c]$ denote the second largest number among a, b, c. For example, $s[1, 10, 3] = 3$, $s[5, 3, 5] = 5$, and $s[2, 2, 2] = 2$. Let $f_1(x)$, $f_2(x)$, $f_3(x)$ be three measurable functions on a set E on the real line. Prove that the function

$$g(x) = s[f_1(x), f_2(x), f_3(x)]$$

is measurable as well.

Problem 3. Let F be a bounded, closed set on the line, and

$$O_n = \bigcup_{x \in F} \left(x - \frac{1}{n}, x + \frac{1}{n} \right).$$

Prove that

$$\lim_{n \to \infty} mO_n = mF,$$

where m is the Lebesgue measure.

Problem 4. Prove that there is a countable collection $\{E_1, E_2, \ldots\}$ of measurable sets on $[0, 1]$ such that for any measurable set E on $[0, 1]$ and any $\varepsilon > 0$ there exists a set E_n from the collection such that

$$m(E \,\Delta\, E_n) \le \varepsilon,$$

where m is the Lebesgue measure and $E \,\Delta\, E_n = (E \setminus E_n) \cup (E_n \setminus E)$.

Problem 5. Let us enumerate all rational points on $[0, 1]$,

$$\mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}.$$

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\theta(x - r_n)}{n^{3/2}}$$

on [0,1], where $\theta(x)$ is the Heaviside step function, $\theta(x) = \chi_{[0,\infty)}(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0. \end{cases}$

Prove that $f'(x) = 0$ almost everywhere.

Problem 6. Prove that if f_n converges to f in $L^4[E]$ and g_n converges to g in $L^6[E]$, where $mE < \infty$, then $f_n g_n$ converges to fg in $L^2[E]$.

QUALIFYING EXAM IN REAL ANALYSIS
IUPUI, AUGUST 2014

PAVEL BLEHER

Problem 1. Let A, B and C be measurable sets on the line such that

$$m(A \,\Delta\, B) + m(B \,\Delta\, C) = m(A \,\Delta\, C)\,,$$

where m is the Lebesgue measure and as usual,

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A).$$

Prove that

$$m[B \setminus (A \cup C)] = 0\,.$$

Problem 2. Let E be a measurable set on the line and f a function on E such that the set

$$\big\{x \in E \,\big|\, f(x) < a \sin x\big\}$$

is measurable for any real a. Prove that the function f is measurable.

Problem 3. Let us enumerate all rational numbers on $[0,1]$, $\mathbb{Q} \cap [0,1] = \{r_1, r_2, \ldots\}$. Prove that the function

$$f(x) = \sum_{k=1}^{\infty} \frac{|x - r_k|^{1/2}}{k^{3/2}}$$

is differentiable almost everywhere on $[0,1]$.

Problem 4. Let $f \in L^2[0,1]$. Prove that

$$\left[\int_0^1 \left| f(x) - \int_0^1 f(x)dx \right| dx\right]^2 \le \int_0^1 f^2(x)dx - \left(\int_0^1 f(x)dx\right)^2.$$

Problem 5. Let $T(f)$ be a bounded linear functional on $L^2[0,1]$ such that $T(x^n) = \frac{1}{n+1}$ for $n = 0, 1, 2, \ldots$. Prove that

$$T(f) = \int_{[0,1]} f(x)\, dx$$

for all $f \in L^2[0,1]$.

Problem 6. Prove that the sequence of functions,

$$f_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k^{5/6}}\,, \quad n = 1, 2, \ldots,$$

is a Cauchy sequence in $L^2[0, 2\pi]$.

IUPUI Qualifying Exam in Real Analysis, Winter 2013

Pavel Bleher

Problem 1. Let A, B and C be measurable sets on the line such that

$$m(A \cap B) = 0.9, \quad m(A \cap C) = 0.9, \quad m(B \cap C) = 0.9, \quad m(A \cup B \cup C) = 1.$$

Prove that

$$m(A \cap B \cap C) \geq 0.85.$$

Problem 2. Suppose that a sequence $E_1, E_2, \ldots$ of measurable sets on $[0, 1]$ satisfies the Cauchy condition, so that for every $\varepsilon > 0$ there exists $N > 0$ such that

$$m(E_n \,\Delta\, E_m) \leq \varepsilon,$$

for all $n, m \geq N$, where $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$. Prove that there exists a measurable set E on $[0, 1]$ such that

$$\lim_{n\to\infty} m(E_n \,\Delta\, E) = 0.$$

Problem 3. Evaluate the limit,

$$\lim_{n\to\infty} \int_0^1 \frac{nx^n dx}{(1 + x^n)^2},$$

and justify your answer.

Problem 4. Let us enumerate all rational points on $[0, 1]$,

$$\mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}.$$

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{|x - r_n|}{n^{3/2}}$$

on [0,1]. Prove that the function $f(x)$ is differentiable almost everywhere.

Problem 5. Prove that if f_n converges to f in $L^p(E)$ and g_n converges to g in $L^q(E)$, where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \qquad p > 0, \quad q > 0, \quad r \geq 1,$$

then $f_n g_n$ converges to fg in $L^r(E)$.

Problem 6. Let

$$f_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k^{2/3}}.$$

Prove that there exists an integrable function $f(x)$ on the interval $[0, 2\pi]$ such that

$$\lim_{n\to\infty} \int_{[0,2\pi]} |f(x) - f_n(x)| dx = 0.$$

QUALIFYING EXAM IN REAL ANALYSIS

IUPUI, AUGUST 2013

PAVEL BLEHER

Problem 1. Let E be a set on the line of a finite outer measure, $m^*E < \infty$, and

$$S = \sup_{\text{measurable } A,\ A \subset E} mA\,.$$

Prove that $S = m^*E$ if and only if E is measurable.

Problem 2. Prove that if $f(x)$ is an absolutely continuous function on the segment $[0,1]$ such that $f'(x) = 1$ almost everywhere, then $f(x) = x + C$, where C is a constant.

Problem 3. Let $\{f_n(x),\ n = 1, 2, \ldots\}$ be a sequence of integrable functions on the segment $[0,1]$ such that

$$\sum_{n=1}^{\infty} \|f_n\|_{L^1[0,1]} < \infty\,.$$

Prove that the series

$$\sum_{n=1}^{\infty} \sin f_n(x)$$

converges in $L^1[0,1]$.

Problem 4. Let $F(f)$ be a bounded linear functional on $L^1[0,1]$ such that $F(f) = 0$ for all f such that

$$\int_0^1 f(x)dx = 0\,.$$

Prove that then there is a real number C such that for all $f \in L^1[0,1]$,

$$F(f) = C \int_0^1 f(x)dx\,.$$

Problem 5. Prove that if f_n converges to f in $L^3[0,1]$, then $(f_n)^2$ converges to f^2 in $L^1[0,1]$.

Problem 6. Let f be an integrable function on the segment $[0,1]$ such that for any $0 \le a < b \le 1$,

$$\left| \int_a^b f(x)dx \right| \le (b-a)^2.$$

Prove that $f = 0$ almost everywhere.

IUPUI QUALIFYING EXAM MATH 544
WINTER 2012

PAVEL BLEHER

Problem 1. Prove that if A is a set on $[0,1]$ and A_n its $(1/n)$-neighborhood, i.e.,

$$A_n = \left\{ x \in \mathbb{R} : \; \exists y \in A, \; |x-y| < \frac{1}{n} \right\},$$

then

$$\bigcap_{n=1}^{\infty} A_n = \overline{A},$$

where $\overline{A}$ is the closure of A.

Problem 2. Prove that the collection of all measurable sets on $[0,1]$ is separable, so that there exists a countable collection $\mathcal{M}_0 = \{E_1, E_2, \dots\}$ of measurable sets on $[0,1]$ such that for any measurable set E on $[0,1]$ and any $\varepsilon > 0$ there exists a set $E_n \in \mathcal{M}_0$ such that

$$m\big(E \,\Delta\, E_n\big) \le \varepsilon,$$

where $E \,\Delta\, E_n = (E \setminus E_n) \cup (E_n \setminus E)$.

Problem 3. Let us enumerate all rational points on $[0,1]$,

$$\mathbb{Q} \cap [0,1] = \{r_1, r_2, \dots\}.$$

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 |x - r_n|^{1/2}}$$

on [0,1]. Prove that the series defining the function $f(x)$ converges for almost all x and f is integrable.

Problem 4. Prove that if a sequence $\langle f_n \rangle$ of integrable functions on a measurable set E converges to a function f in such a way that

$$\int_E |f_n(x) - f(x)| \, dx \le \frac{1}{2^n}, \qquad n = 1, 2, \dots,$$

then $f_n(x)$ converges to $f(x)$ a.e. on E.

Problem 5. Prove that if $f \in L^2[0,1]$ and

$$\int_0^1 f(x) \, dx = 0,$$

then

$$\int_0^1 |f(x) - x|^2 dx \ge \frac{1}{4}.$$

Problem 6. Let $f(x)$ be an absolutely continuous function on an interval $[a, b]$ such that $f'(x) \in L^2[a, b]$. Prove that there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^{1/2}$$

for all $x, y \in [a, b]$.

QUALIFYING EXAM IN REAL ANALYSIS

IUPUI, AUGUST 2012

PAVEL BLEHER

Problem 1. Let f be a continuously differentiable function on $[0, 2\pi]$ such that $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$. Prove that for any $\varepsilon > 0$, there is a trigonometric polynomial,

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \big[a_k \sin(kx) + b_k \cos(kx)\big],$$

such that

$$\sup_{0 \le x \le 2\pi} |f(x) - g(x)| + \sup_{0 \le x \le 2\pi} |f'(x) - g'(x)| \le \varepsilon.$$

Problem 2. Prove that for any Lebesgue measurable set E on $[0, 1]$ and any $\varepsilon > 0$, there is a finite collection of disjoint intervals (a_j, b_j), $1 \le j \le n$, on $[0, 1]$ such that all the end-points $\{a_j, b_j\}$ are rational and

$$m\left(E\,\Delta \bigsqcup_{j=1}^{n} (a_j, b_j)\right) \le \varepsilon,$$

where $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Problem 3. Let f satisfy a Lipschitz condition on $[0, 1]$, so that there is a constant $M > 0$ such that $|f(x) - f(y)| \le M|x - y|$ for all x and y on $[0, 1]$. Prove that f is differentiable almost everywhere on $[0, 1]$.

Problem 4. Let $f \in L^2[0, 1]$. Prove that

$$\inf_{-\infty < a < \infty} \int_0^1 |f(x) - a|dx \le \sqrt{\int_0^1 f^2(x)dx - \left(\int_0^1 f(x)dx\right)^2}.$$

Problem 5. The *Sierpinski carpet* is a set on the plane which is constructed as follows. We begin with a square. The square is cut into 9 congruent subsquares in a 3-by-3 grid, and the open central subsquare is removed. The same procedure is then applied recursively to the remaining 8 subsquares, ad infinitum. Prove that the planar Lebesgue measure of the Sierpinski carpet is equal to 0.

Problem 6. Prove that for any $\infty > p \ge 1$ there is a constant $C_p > 0$ such that for any integrable function f on $[0, 1]$,

$$\|\ln(1 + |f|)\|_{L^p[0,1]} \le C_p \left(1 + \|f\|_{L^1[0,1]}\right).$$

Qualifying Examination – Math 54400 – January 2011

1. Assume that $B_n \subset A_n \subset \mathbb{R}$ for $n = 1, 2, \dots$ are measurable sets. Prove that

$$m\left(\bigcup_{n=1}^{\infty} A_n \setminus \bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty}(m(A_n) - m(B_n)).$$

2. Let A_1, A_2, A_3, A_4 be measurable subsets of $[0, 1]$, such that

$$\sum_{k=1}^{4} m(A_k) > 3.$$

Prove that

$$m\left(\bigcap_{k=1}^{4} A_k\right) > 0.$$

3. Consider a sequence of functions $f_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, convergent to f in $L^1(\mathbb{R})$ and to g in $L^2(\mathbb{R})$. Prove that $f = g$ a.e.

4. Let E be a measurable subset of $\mathbb{R}$ and $f \in L^1(E)$. Set $A_n = \{x \in E : |f(x)| \geq n\}$ for $n = 1, 2, \dots$. Prove that

$$\sum_{n=1}^{\infty} m(A_n) \leq \int_E |f|.$$

5. Let $p, q, r \in (1, \infty)$ with $1/p + 1/q + 1/r = 1$. Prove that for every functions $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ and $h \in L^r(\mathbb{R})$

$$\int_{\mathbb{R}} |fgh| \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r.$$

6. Let $f \in L^1([0, 1])$. Prove that $\sqrt{1 + f^2} \in L^1([0, 1])$ and

$$\sqrt{1 + (\|f\|_1)^2} \leq \left\|\sqrt{1 + f^2}\right\|_1.$$

Hint: For inequality, use Jensen's inequality.

Math 54400 – Qualifying Examination – August 2011

1. Prove that for every $\varepsilon > 0$ there exists an open set $A \subset \mathbb{R}$, dense in $\mathbb{R}$, of Lebesgue measure less than ε.

2. Prove that if a function $f : [0,1] \to \mathbb{R}$ has bounded variation and $\inf\{|f(x)| : x \in [0,1]\} > 0$, then $1/f$ also has bounded variation.

3. Let $a < b$ be real numbers. Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function with bounded derivative and such that $f(a) = 0$. Prove that

$$\sup\{|f(x)| : x \in [a,b]\} \leq \sqrt{(b-a) \int_a^b (f'(x))^2 \, dx}.$$

4. Give example of a function $f : [0,1] \to \mathbb{R}$ which is integrable, but is not essentially bounded on any subinterval of $[0,1]$.

5. Show that $f \mapsto \sqrt{\|f\|_1 \cdot \|f\|_\infty}$ is not a norm in $L^\infty([0,1])$.

6. For $n = 1, 2, 3, \ldots$ define a function $f_n : [0,1) \to \mathbb{R}$ as follows. Let $k(x,n) = \lfloor 2^n x \rfloor$ (the largest integer smaller than or equal to $2^n x$); let

$$f_n(x) = \begin{cases} 0 & \text{if } k(x,n) \text{ is even,} \\ 1 & \text{if } k(x,n) \text{ is odd.} \end{cases}$$

Does there exist a subsequence of the sequence $(f_n)_{n=1}^\infty$ convergent almost everywhere? Explain.

IUPUI QUALIFYING EXAM ON REAL ANALYSIS

JANUARY 2010

PAVEL BLEHER

Problem 1. Consider the space l of sequences $x = (x_1, x_2, \ldots)$ such that the limit, $\lim_{n\to\infty} x_n$, exists. Define the norm in l as the l^∞-norm,

$$\|x\|_\infty = \sup_n |x_n|.$$

Is the space l complete or not? Explain.

Problem 2. Prove that the linear functional

$$F[f] = \int_0^1 x f(x)\, dx$$

on $L^3[0,1]$ is bounded and evaluate its norm.

Problem 3. Let $f \in L^1(-\infty,\infty)$. Find the limit,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \left(\cos x + \frac{\sin^3 x}{2} \right)^n dx,$$

and justify your answer.

Problem 4. Let C be the Cantor ternary set,

$$C = [0,1] \setminus \left[\left(\frac{1}{3}, \frac{2}{3} \right) \cup \left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \cup \ldots \right],$$

and let

$$E = \{x + y \mid x \in C,\ y \in \mathbb{Q}\},$$

where $\mathbb{Q}$ is the set of rational numbers. Find the Lebesgue measure of E and justify your answer.

Problem 5. Prove that if $f \in L^1(-\infty,\infty)$ and

$$g(x) = \int_{-\infty}^{\infty} f(x-y) e^{-|y|} dy,$$

then $g \in L^1(-\infty,\infty)$ and

$$\|g\|_1 \le 2\|f\|_1.$$

Problem 6. Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that if $f \in L^6[X]$ and $g \in L^3[X]$, then $fg \in L^2[X]$.

IUPUI QUALIFYING EXAM ON REAL ANALYSIS

SUMMER 2010

PAVEL BLEHER

Problem 1. The Riemann function $R(x)$ on the segment $[0,1]$ is defined as

$$R(x) = \begin{cases} 0, & \text{when } x \text{ is irrational or } x = 0, \\ \dfrac{1}{q}\,, & \text{when } x = \dfrac{p}{q} > 0\,, \text{ where } p \text{ and } q \text{ are relatively prime.} \end{cases}$$

Is there a sequence $\langle f_n \rangle$ of continuous functions on $[0,1]$ which converges to $R(x)$ at every point $x \in [0,1]$? Explain.

Problem 2. Let us enumerate all rational points on the segment $[0,1]$,

$$\mathbb{Q} \cap [0,1] = \{r_1, r_2, \ldots\}.$$

Define the function

$$f(x) = \sum_{n:\, r_n < x} \frac{1}{2^n}\,.$$

Prove that

(1) the function $f(x)$ is discontinuous at every rational point,
(2) the function $f(x)$ is continuous at every irrational point,
(3) the function $f(x)$ is differentiable almost everywhere,
(4) $f'(x) = 0$ almost everywhere.

Problem 3. Let $E_1 \subset E_2 \subset \ldots$ be an increasing sequence of Lebesgue measurable sets on the line, such that the set $E = \cup_{n=1}^{\infty} E_n$ has a finite Lebesgue measure. Prove that for any set $A \subset \mathbb{R}$,

$$\lim_{n \to \infty} m^*(A \cap E_n) = m^*\left(A \cap E\right).$$

Problem 4. Let $f \in L^1(-\infty, \infty)$. Find the limit,

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{f(x) e^{nx} dx}{1 + e^{nx}}\,,$$

and justify your answer.

Problem 5. Consider the functional

$$F(f) = \int_0^1 e^x f(x) dx.$$

Prove that F is a linear bounded functional on $L^4[0,1]$ and calculate the norm of F on $L^4[0,1]$.

Problem 6. Prove that if $f, g \in L^1(-\infty, \infty)$ and

$$h(x) = \int_{-\infty}^{\infty} f(x - e^y) g(y) dy,$$

then $h \in L^1(-\infty, \infty)$ and

$$\|h\|_1 \leq \|f\|_1 \|g\|_1.$$

Qualifying Exam in Real Analysis

IUPUI, January 2009

Problem 1. Does there exist a measurable set of non-zero measure which is closed and has no interior points?

Problem 2. Let $\langle f_n \rangle$ be a sequence of measurable functions on $(0, 1)$. Show that the set

$$E = \{x \in (0,1) \mid \langle f_n(x) \rangle \text{ is a convergent sequence}\}$$

is measurable.

Problem 3. Enumerate all rational points on [0,1], so that

$$\mathbb{Q} \cap [0,1] = \{r_1, r_2, \dots\}.$$

Define the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \theta(x - r_n), \qquad 0 \le x \le 1,$$

where

$$\theta(x) = \begin{cases} 1, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Prove that there exists a point $0 < x_0 < 1$ such that $f(x)$ is differentiable at x_0.

Problem 4. Let $g(x)$ be a continious periodic function of period 1 on $\mathbb{R}$. Prove that for any integrable function $f(x)$ on $[0, 1]$,

$$\lim_{n \to \infty} \int_0^1 f(x) g(nx)\, dx = \int_0^1 f(x)\, dx \int_0^1 g(x)\, dx$$

Problem 5. Prove that the operator

$$A: \ f(x) \ \to \ \int_{-\infty}^{+\infty} f(x - y) e^{-y^2}\, dy$$

is a bounded operator in $L^2(-\infty, \infty)$.

Problem 6. Let (X, d) be a compact metric space. Let $\langle f_n \rangle$ be a sequence of continious real valued functions on X such that $f_n(x) \le f_{n+1}(x)$ for all n, x. Assume that the sequence $\langle f_n \rangle$ is pointwise convergent and let

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Show that $f(x)$ is continious if and only if the sequence $\langle f_n \rangle$ converges uniformly.

QUALIFYING EXAM IN REAL ANALYSIS

IUPUI, AUGUST 2009

Problem 1. Prove that the space $C^1[0,1]$ of continuously differentiable functions on $[0,1]$, with the norm,

$$\|f(x)\|_{C^1[0,1]} = \sup_{0\le x\le 1} (|f(x)| + |f'(x)|),$$

is separable.

Problem 2. Find a linear function $ax+b$ such that the norm,

$$\|ax + b - e^x\|_{L^2[0,1]}\,,$$

is minimal.

Problem 3. Let $f(x)$ be a continuous function on $\mathbb{R}$, and let $g(x)$ be a measurable function on $[0,1]$. Is the function $f(g(x))$ measurable? Explain.

Problem 4. Is there an integrable function $f(x)$ on $[0,1]$ such that for any $0 < a < b < 1$,

$$\|f(x)\|_{L^2[a,b]} = \infty.$$

Explain.

Problem 5. Is there an open connected set in $\mathbb{R}^2$ of finite Lebesgue measure, which has a non-empty intersection with any straight line? Explain.

Problem 6. Prove that there is a constant $C > 0$ such that for any continuous function f on $[0,1]$,

$$\|\ln(1+|f|)\|_{L^4[0,1]} \le C\|f\|_{L^2[0,1]}\,.$$

IUPUI QUALIFYING EXAM MATH 544
WINTER 2008

PAVEL BLEHER AND MICHAL MISIUREWICZ

Problem 1. Prove that any set X on the line is separable, i.e., there exists an at most countable set $A \subset X$ whose closure contains X.

Problem 2. Is there a measurable set E on $[0, 1]$ such that for any $0 \le a < b \le 1$,

$$m(E \cap [a, b]) = \frac{b - a}{2},$$

where m stands for the Lebesgue measure?

Problem 3. Prove that if f is integrable on $[0, 1]$, then

$$\lim_{n\to\infty} n \int_0^1 \sin(x/n)\, f(x)\, dx = \int_0^1 x f(x)\, dx.$$

Problem 4. Prove that if functions f_1, f_2 are absolutely continuous on $[0, 1]$, then the function $f(x) = \max\{f_1(x), f_2(x)\}$ is absolutely continuous as well.

Problem 5. Let $f_1, f_2, \dots$ be a sequence of nondecreasing absolutely continuous functions on the closed interval $[0, 1]$ such that the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges at every point $x \in [0, 1]$. Prove that the function f is absolutely continuous.

Hint. Consider the series of the derivatives,

$$\sum_{n=1}^{\infty} f_n'(x),$$

and prove that it converges a.e. to an integrable function.

Problem 6. Suppose that $f \in L^3[0, 1]$. Prove that

$$\|x f(x)\|_2 \le \frac{\|f(x)\|_3}{7^{1/6}}.$$

Hint. Use the Hölder inequality.

IUPUI QUALIFYING EXAM MATH 544
AUGUST 2008

PAVEL BLEHER AND MICHAL MISIUREWICZ

Problem 1. Suppose that $A \subset \mathbb{R}$ is a measurable set. Prove that the set

$$B = A \cup (-A), \qquad -A \equiv \{x: \ -x \in A\},$$

is measurable and

$$mA \leq mB \leq 2mA,$$

where m is the Lebesgue measure.

Problem 2. Prove that for any integrable function f on $[a, b]$,

$$\lim_{n\to\infty} \int_a^b f(x) \sin^3(nx)\, dx = 0.$$

Problem 3. Prove that there is no function $f(x) \in L^\infty[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 [f(x)]^n\, dx = 2.$$

Problem 4. For a given $\varepsilon > 0$, construct an open dense set O on $[0, 1]$ such that $mO < \varepsilon$, where m is the Lebesgue measure.

Problem 5. Prove that if $f \in L^2[0, 1]$ and $g \in L^3[0, 1]$ then $fg \in L^{6/5}[0, 1]$.

Problem 6. Prove the existence of the limit,

$$\lim_{n\to\infty} \int_0^\infty \frac{1 + x^n}{2 + x^n}\, e^{-x}\, dx,$$

and find it.

IUPUI QUALIFYING EXAM MATH 544
SUMMER 2007

PAVEL BLEHER AND MICHAL MISIUREWICZ

Problem 1. Let F be a closed subset of $[0,1]$ and F_ε, where $\varepsilon > 0$, its ε-neighborhood, i.e.,

$$F_\varepsilon = \{x \in \mathbb{R} : \exists\, y \in F,\ |x-y| < \varepsilon\}.$$

Prove that

$$\lim_{\varepsilon \to 0} mF_\varepsilon = mF,$$

where m is the Lebesgue measure. Does this statement hold in the case when F is an arbitrary open subset of $[0,1]$? Explain.

Problem 2. Let A and B be two bounded closed subsets of $\mathbb{R}$. Prove that the set

$$A + B = \{x + y : \ x \in A,\ y \in B\}.$$

is closed and

$$m(A+B) \geq mA + mB,$$

where m is the Lebesgue measure.

Problem 3. Let $f_n : \mathbb{R} \to \mathbb{R}^+$ be a sequence of nonnegative integrable functions on the line such that

$$\lim_{n \to \infty} f_n(x) = 0$$

almost everywhere. Prove that

$$\lim_{n \to \infty} \int_{\mathbb{R}} \min_{1 \leq k \leq n} f_k(x)\, dx = 0.$$

Problem 4. Let f be a nonnegative integrable function on the interval $[0,1]$ such that

$$\int_0^1 f(x)\, dx = 1,$$

and let $g(x) = \sqrt{f(1-x)}$. Prove that the function g is integrable on $[0,1]$ and

$$\int_0^1 g(x)\, dx \leq 1.$$

Problem 5. Let f be an integrable function on $[0,1]$ such that for any $0 \leq a < b \leq 1$,

$$\int_a^{(a+b)/2} f(x)\, dx = \int_{(a+b)/2}^b f(x)\, dx.$$

Prove that then the function f is constant almost everywhere.

Problem 6. Let f be a real valued function on the interval $(0, 1)$. A point x is called a point of discontinuity of f of the *first kind*, if the limits of f at x both from the right and from the left exist and they are not equal. Prove that the set of points of discontinuity of f of the first kind is at most countable.

Hint. Prove first that the set of points of discontinuity of f of the first kind, with the difference of the right and the left limits bigger then $1/n$, is at most countable for any $n = 1, 2, \ldots$.

IUPUI QUALIFYING EXAM MATH 544
SUMMER 2006

PAVEL BLEHER

Problem 1. Suppose that $A \subset [0,1]$ is a measurable set. Prove that the set

$$B = \cos A \equiv \{\cos x,\ x \in A\}$$

is measurable and $mB \le 0.85 mA$, where m is the Lebesgue measure.

Hint. Use that $\sin 1 = 0.841\ldots$

Problem 2. Prove that for any integrable function f on $[a,b]$,

$$\lim_{n\to\infty} \int_a^b f(x)\sin^2(nx)dx = \frac{1}{2}\int_a^b f(x)dx.$$

Problem 3. Prove that there is no positive integrable function $f(x)$ on $[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 [f(x)]^n\, dx = 2.$$

Problem 4. Let A be a measurable set on $[0,1]$. A point $x \in [0,1]$ is called a *density point* of A if

$$\lim_{\varepsilon\to 0} \frac{m(A \cap [x, x+\varepsilon])}{|\varepsilon|} = 1,$$

where m stands for the Lebesgue measure. Prove that almost all points of the set A are its density points.

Hint. Apply the theorem on the differentiation of an integral.

Problem 5. Let $f \in L^p(X)$, where $(X, \mathcal{A}, \mu)$ is a finite measure space, $\mu(X) < \infty$, and let $p > r \ge 1$. Prove that then $f \in L^r(X)$ and

$$\|f\|_r \le (\mu X)^{\frac{1}{r}-\frac{1}{p}} \|f\|_p.$$

Problem 6. Prove that there exists the limit,

$$\lim_{n\to\infty} \int_0^1 \frac{e^{-x}\cos x}{nx^2 + \frac{1}{n}}\, dx,$$

and find it.

Hint. Make a change of variable.