Problem 1. Let $E$ be the set of real numbers $x$ on the interval $[0, 1]$ such that in the
decimal form of $x = 0.i_1i_2i_3\ldots$ there is no string of four consecutive digits 2018. Prove that
(1) the set $E$ is uncountable;
(2) $mE=0$, where $mE$ is the Lebesgue measure of $E$.

Problem 2. Prove that the function
\[ f(x) = \sum_{n=1}^{\infty} \frac{|x - n^{-1}|^{1/2}}{n^2} \]
is absolutely continuous on $[0, 1]$.

Problem 3. Prove that for any integrable function $f$ on the interval $[a, b]$,
\[ \lim_{n \to \infty} \int_{a}^{b} f(x) \sin^4(nx) \, dx = \frac{3}{8} \int_{a}^{b} f(x) \, dx. \]

Problem 4. Let $f(x) \in L^2(-\infty, \infty)$. Prove that
\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} f(x)f(x+n) \, dx = 0. \]

Problem 5. Let $f \in L^3[0, \pi]$ and
\[ g(x) = \frac{f(x)}{|\sin x|^{0.1}}. \]
Prove that $\|g\|_2 \leq 2\|f\|_3$.

Problem 6. Let $f(x)$ be an integrable function on $(-\infty, \infty)$ and $n \geq 1$ an integer. Define
\[ f_n(x) = n \int_{x}^{x+n^{-1}} f(t) \, dt. \]
Prove that $\|f_n\|_1 \leq \|f\|_1$. 
Problem 1. A real number \( x \) on \([0,1]\) is called a Liouville type number, if its decimal representation has the form
\[
x = \sum_{j=1}^{\infty} \frac{1}{10^{n_j}},
\]
where \( n_j \in \mathbb{N} \) are such that
\[
n_{j+1} - n_j \geq j!, \quad j = 1, 2, \ldots.
\]
(Liouville proved that all such numbers are transcendental.) Prove that the set of Liouville type numbers is uncountable.

Problem 2. Prove that the series
\[
\sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{|\sin(nx)|}}
\]
converges almost everywhere on \([0, 2\pi]\).

Problem 3. Let \( f(x) \) be an absolutely continuous function on an interval \([a, b]\) such that \( f'(x) \in L^3[a, b] \). Prove that there exists a constant \( C > 0 \) such that
\[
|f(y) - f(x)| \leq C|y - x|^{2/3}
\]
for all \( a \leq x < y \leq b \).

Problem 4. Let \( f(x) \in L^1(-\infty, \infty) \). Prove that
\[
\lim_{a \to \infty} \int_{-\infty}^{\infty} \frac{f(x) \, dx}{1 + (x - a)^2} = 0.
\]

Problem 5. Let \( f \in L^6[0, 1], g \in L^7[0, 1], \) and \( h \in L^8[0, 1] \). Prove that \( fgh \in L^2[0, 1] \).

Problem 6. Let \( E_1, E_2, \ldots \) be a sequence of measurable sets on the line such that \( m(E_j \triangle E_k) < \infty \) for all \( j, k = 1, 2, \ldots \) and
\[
\lim_{j,k \to \infty} m(E_j \triangle E_k) = 0, \quad A \triangle B := (A \setminus B) \cup (B \setminus A).
\]
Prove that there exists a measurable set \( E \) such that
\[
\lim_{j \to \infty} m(E_j \triangle E) = 0.
\]
Problem 1. A real number is called transcendental if it is not a root of a polynomial with integer coefficients. Let $E$ be the set of transcendental numbers on the interval $[0, 1]$. Prove that the set $E$ is measurable and $mE = 1$.

Problem 2. Let $A$ and $E$ be disjoint sets on the line and let $E$ be measurable. Prove that

$$
m^*(A \cup E) = m^*A + mE.
$$

Problem 3. Prove that for any integrable function $f$ on the interval $[a, b]$,

$$
\lim_{n \to \infty} \int_a^b f(x) |\sin(nx)| \, dx = \frac{2}{\pi} \int_a^b f(x) \, dx.
$$

Problem 4. Let $f(x) \in L^1(-\infty, \infty)$. Prove that

$$
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x - n) \arctan \left( \frac{x}{n} \right) \, dx = \frac{\pi}{4} \int_{-\infty}^{\infty} f(x) \, dx.
$$

Problem 5. Let $f(x)$ be a nonnegative integrable function on the interval $[a, b]$. Prove that

$$
\left( \int_a^b f(x) \sin x \, dx \right)^2 + \left( \int_a^b f(x) \cos x \, dx \right)^2 \leq \left( \int_a^b f(x) \, dx \right)^2.
$$

Problem 6. Let $f(x) \in L^2[0, 1]$. Define

$$
g(x) = \frac{1}{x^{4/3}} \int_0^x f(t) \, dt.
$$

Prove that $\|g\|_1 \leq 6\|f\|_2$. 
Problem 1. Let $A$ and $B$ be two sets on the interval $[0, 1]$ such that
$$\overline{A} \cap \overline{B} = \emptyset,$$
where $\overline{A}$ and $\overline{B}$ are the closures of the sets $A$ and $B$, respectively, and $\emptyset$ is the empty set. Prove that
$$m^*(A \cup B) = m^* A + m^* B,$$
where $m^*$ is the outer measure.

Problem 2. Let $f(x)$ be an absolutely continuous function on $[0, 1]$ and $E$ a set on $[0, 1]$ such that $mE = 0$, where $mE$ is the Lebesgue measure of the set $E$. Let $f(E)$ be the image of the set $E$ with respect to $f$,
$$f(E) = \{y \in \mathbb{R} \mid \exists x \in [0, 1] \text{ such that } f(x) = y\}.$$  
Prove that $mf(E) = 0$.

Problem 3. Consider the set $E$ of numbers $x$ on $[0, 1]$ such that in every 100 consecutive digits $i_{k+1}, \ldots, i_{k+100}$, $k = 0, 1, 2, \ldots$, of a decimal fraction representing $x$, $x = 0.i_1i_2i_3\ldots$, where $i_k \in \{0, 1, \ldots, 9\}$, there is at least one 0. Prove that $mE = 0$.

Problem 4. Let $f(x)$ be an integrable function on the interval $[a, b]$. Define recursively the functions,
$$f_0(x) = f(x), \quad f_k(x) = \int_a^x f_{k-1}(t) \, dt, \quad k \geq 1; \quad a \leq x \leq b.$$  
Prove that the series
$$F(x) = \sum_{k=1}^{\infty} f_k(x)$$
converges uniformly on $[a, b]$ and $F(x)$ is an absolutely continuous function.

Problem 5. Let $A$ be a measurable subset of $[0, 2\pi]$. Prove that
$$\lim_{n \to \infty} \int_A \cos^2(nx) \, dx = \frac{1}{2} mA,$$
where $mA$ is the measure of $A$.

Problem 6. Let $f(x)$ be an integrable function on $[0, \infty)$. Prove that
$$\lim_{n \to \infty} n \int_0^1 x f(nx) \, dx = 0.$$
Problem 1: Show that there is a compact set $K \subset [0, 1]$ with Lebesgue measure $m(K) > 0$, but such that $K$ contains no non-empty open interval.

Problem 2: Suppose $(E_n)$ is a sequence of measurable subsets of $[0, 1]$ with Lebesgue measure $m(E_n \Delta E_{n+1}) \leq 1/2^n$ for all $n \geq 1$. Here $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

(a) Show that $(1_{E_n})$ is a Cauchy sequence in $L_1$-norm in $L_1[0,1]$.

(b) Show that there is a measurable set $E$ such that $m(E \Delta E_n) \to 0$ as $n \to \infty$.

Problem 3: Let $f \geq 0$ be a measurable function on $[0, 1]$. Show that if $\sum_{n=1}^{\infty} m\{f \geq n^2\} < \infty$, then $\int_0^1 \sqrt{f(x)} \, dm(x) < \infty$.

Problem 4: Describe a measurable function $f$ on $[0, 1]$ such that $f \in L_1[0,1]$, but for all $p, 1 < p \leq \infty$, $f \notin L_p[0,1]$.

Problem 5: Suppose $h : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^1 h(x) \, dx = 0$. For $x \geq 0$, let $\{x\}$ be the fractional part of $x$: $\{x\} = x - p$ where $p$ is a positive integer such that $p \leq x < p+1$.

(a) Show that for any $0 \leq a \leq b \leq 1$, we have $\int_a^b h(\{nx\}) \, dx \to 0$ as $n \to \infty$.

(b) Using a), show that for any $g \in L_1[0,1]$, we have $\int_0^1 h(\{nx\}) g(x) \, dx \to 0$ as $n \to \infty$.

Problem 6: Show that for $f \in L_1(\mathbb{R})$ and any $\epsilon > 0$, we have

$$\lim_{n \to \infty} m\{x \in \mathbb{R} : |f(x + \frac{1}{n}) - f(x)| \geq \epsilon\} = 0.$$
Throughout this examination, $m$ denotes the Lebesgue measure on $\mathbb{R}$.

**Problem 1:** Show that there is a compact set $K \subset [0, 1]$ with $m(K) \geq 11/12$, but such that $K$ contains no non-empty open intervals.

**Problem 2:** Suppose $(f_n)$ is a sequence in $L_1[0, 1]$ such that $\lim_{n \to \infty} \int_0^1 |f_n(x)| \, dm(x) = 0$. Show that for all $\delta > 0$, we have $m \{ x \in [0, 1] : |f_n(x)| \geq \delta \} \to 0$ as $n \to \infty$.

**Problem 3:** Let $f \geq 0$ be a measurable function on $[0, 1]$. Assume $m \{ f \geq x \} \leq 1/x^2$ for all $x \geq 1$. Show that $f \in L_r[0, 1]$ for all $1 \leq r < 2$.

**Problem 4:** Show that for $1 < p \leq \infty$, $L_p[0, 1] \subset L_1[0, 1]$ but $L_p(\mathbb{R})$ is not a subset of $L_1(\mathbb{R})$.

**Problem 5:** Let $(r_n)$ be the sequence of functions such that $r_n : [0, 1) \to \{-1, 1\}$ given by the rule that $r_n(x) = (-1)^k$ for $x \in [k/2^n, (k+1)/2^n)$ with $k = 0, \ldots, 2^n - 1$. Let $f \in L_1[0, 1]$. Show that $\lim_{n \to \infty} \int_0^1 f(x)r_n(x) \, dm(x) = 0$.

**Problem 6:** Show that for $f \in L_1(\mathbb{R})$, we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} |f(x + 1/n) - f(x)| \, dm(x) = 0.$$
Problem 1. Let $E \subset [0, 1]$ be a measurable set such that there exists $\varepsilon > 0$ such that
\[m(E \cap [a, b]) \geq \varepsilon |b - a|\]
for all $[a, b] \subset [0, 1]$. Prove that $mE = 1$.

Problem 2. Let $E$ be a measurable set on the line and $f(x)$ a nonnegative measurable function on $E$ such that the limit,
\[L = \lim_{n \to \infty} \int_E [f(x)]^n dx,
\]
exists, where $0 < L < \infty$. Prove that $m\{x \in E \mid f(x) = 1\} = L$.

Problem 3. Let $f$ be an integrable function on a set $E$ of a finite Lebesgue measure $mE < \infty$. Let $E_n = \{x \in E \mid |f(x)| \geq n\}$. Prove that
\[\sum_{n=0}^{\infty} mE_n \geq \int_E |f(x)| dx \geq \sum_{n=1}^{\infty} mE_n.
\]

Problem 4. Prove that if $f$ is absolutely continuous on $[a, b]$ and $g(x) = e^{|f(x)|}$ then $g$ is absolutely continuous on $[a, b]$ as well.

Problem 5. Prove that if $f \in L^1[0, \infty)$, then
\[\lim_{n \to \infty} \frac{1}{n} \int_0^n x f(x) dx = 0.
\]

Problem 6. Prove that if $f(x) \in L^{4+\varepsilon}[0, 1]$, where $\varepsilon > 0$, and $g(x) = f(x^2)$, then $g(x) \in L^2[0, 1]$. 
Problem 1. Let $f$ be a measurable function on $[0, 1]$ such that $f$ is finite almost everywhere. Prove that for any $\varepsilon > 0$, there is a polynomial $P(x) = \sum_{k=0}^{n} p_k x^k$, such that
\[ m\{x : |f(x) - P(x)| \geq \varepsilon\} \leq \varepsilon, \]
where $m$ is the Lebesgue measure.

Problem 2. Let $\mathcal{M}$ be the $\sigma$-algebra of all Lebesgue measurable sets on $[0, 1]$. Prove that there is a countable collection of sets $D \subset \mathcal{M}$ such that for every set $E \in \mathcal{M}$ and every $\varepsilon > 0$ there is a set $A \in D$ such that
\[ m(A \Delta E) \leq \varepsilon, \quad A \Delta E \equiv (A \setminus E) \cup (E \setminus A). \]

Problem 3. Let $f$ be a measurable function on $[0, 1]$ such that $0 \leq f(x) \leq 1$ for all $x$ on $[0, 1]$, and let $g$ be a function of bounded variation on $[0, 1]$. Prove that the function $g(f(x))$ is measurable on $[0, 1]$.

Problem 4. Let $f$ be a continuous function on $[0, 1]$, which is differentiable almost everywhere. Suppose that $f' \in L^p[0, 1]$, where $p > 1$. Let $\alpha = 1 - \frac{1}{p}$. Prove that there is a constant $C > 0$ such that
\[ |f(y) - f(x)| \leq C|y - x|^\alpha, \]
for all $x, y$ on $[0, 1]$.

Problem 5. Let $f \in L^\infty[0, 1]$. Prove that for every $p \geq 1$,
\[ \exp\left[ \int_0^1 f(x) \, dx \right] \leq \|e^{f(x)}\|_p. \]

Problem 6. Prove that for every $p < 1$ the function
\[ f(x) = |\sin (x^{-1})|^{-p} \]
is integrable on $[0, 1]$.
Problem 1. Let $A, B$ and $C$ be measurable sets on the line such that
\[
m(A \cap B) = 0.3, \quad m(A \cap C) = 0.4, \quad m(B \cap C) = 0.2,
\]
\[
m(A \cup B) = 0.8, \quad m(A \cup C) = 0.8, \quad m(B \cup C) = 0.9,
\]
\[
m(A \cup B \cup C) = 1.
\]
where $m$ is the Lebesgue measure. Find $m(A \cap B \cap C)$.

Problem 2. For real numbers $a, b, c$, let $s[a, b, c]$ denote the second largest number among $a, b, c$. For example, $s[1, 10, 3] = 3$, $s[5, 3, 5] = 5$, and $s[2, 2, 2] = 2$. Let $f_1(x), f_2(x), f_3(x)$ be three measurable functions on a set $E$ on the real line. Prove that the function
\[
g(x) = s[f_1(x), f_2(x), f_3(x)]
\]
is measurable as well.

Problem 3. Let $F$ be a bounded, closed set on the line, and
\[
O_n = \bigcup_{x \in F} \left( x - \frac{1}{n}, x + \frac{1}{n} \right).
\]
Prove that
\[
\lim_{n \to \infty} mO_n = mF,
\]
where $m$ is the Lebesgue measure.

Problem 4. Prove that there is a countable collection \{\mathcal{E}_1, \mathcal{E}_2, \ldots\} of measurable sets on $[0, 1]$ such that for any measurable set $E$ on $[0, 1]$ and any $\varepsilon > 0$ there exists a set $\mathcal{E}_n$ from the collection such that
\[
m(E \Delta \mathcal{E}_n) \leq \varepsilon,
\]
where $m$ is the Lebesgue measure and $E \Delta \mathcal{E}_n = (E \setminus \mathcal{E}_n) \cup (\mathcal{E}_n \setminus E)$.

Problem 5. Let us enumerate all rational points on $[0, 1]$,
\[
\mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}.
\]
Consider the function
\[
f(x) = \sum_{n=1}^{\infty} \frac{\theta(x - r_n)}{n^{3/2}}
\]
on $[0, 1]$, where $\theta(x)$ is the Heaviside step function, $\theta(x) = \chi_{[0, \infty)}(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$
Prove that $f'(x) = 0$ almost everywhere.

Problem 6. Prove that if $f_n$ converges to $f$ in $L^4[E]$ and $g_n$ converges to $g$ in $L^6[E]$, where $mE < \infty$, then $f_ng_n$ converges to $fg$ in $L^2[E]$. 

Problem 1. Let $A, B$ and $C$ be measurable sets on the line such that
\[ m(A \Delta B) + m(B \Delta C) = m(A \Delta C), \]
where $m$ is the Lebesgue measure and as usual,
\[ A \Delta B = (A \setminus B) \cup (B \setminus A). \]
Prove that
\[ m[B \setminus (A \cup C)] = 0. \]

Problem 2. Let $E$ be a measurable set on the line and $f$ a function on $E$ such that the set
\[ \{ x \in E \mid f(x) < a \sin x \} \]
is measurable for any real $a$. Prove that the function $f$ is measurable.

Problem 3. Let us enumerate all rational numbers on $[0, 1]$, $\mathbb{Q} \cap [0, 1] = \{ r_1, r_2, \ldots \}$. Prove that the function
\[ f(x) = \sum_{k=1}^{\infty} \frac{|x - r_k|^{1/2}}{k^{3/2}} \]
is differentiable almost everywhere on $[0, 1]$.

Problem 4. Let $f \in L^2[0, 1]$. Prove that
\[ \left[ \int_0^1 |f(x) - \int_0^1 f(x)dx| \, dx \right]^2 \leq \int_0^1 f^2(x)dx - \left( \int_0^1 f(x)dx \right)^2. \]

Problem 5. Let $T(f)$ be a bounded linear functional on $L^2[0, 1]$ such that $T(x^n) = \frac{1}{n^{n+1}}$ for $n = 0, 1, 2, \ldots$. Prove that
\[ T(f) = \int_{[0,1]} f(x) \, dx \]
for all $f \in L^2[0, 1]$.

Problem 6. Prove that the sequence of functions,
\[ f_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k^{3/6}}, \quad n = 1, 2, \ldots, \]
is a Cauchy sequence in $L^2[0, 2\pi]$. 

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[1] LEARNING PHASE

[2] PROBLEM 1

[3] PROBLEM 2

[4] PROBLEM 3

[5] PROBLEM 4

[6] PROBLEM 5

[7] PROBLEM 6
IUPUI Qualifying Exam in Real Analysis, Winter 2013
Pavel Bleher

Problem 1. Let $A, B$ and $C$ be measurable sets on the line such that
\[ m(A \cap B) = 0, \quad m(A \cap C) = 0, \quad m(B \cap C) = 0, \quad m(A \cup B \cup C) = 1. \]
Prove that
\[ m(A \cap B \cap C) \geq 0.85. \]

Problem 2. Suppose that a sequence $E_1, E_2, \ldots$ of measurable sets on $[0, 1]$ satisfies the Cauchy condition, so that for every $\varepsilon > 0$ there exists $N > 0$ such that
\[ m(E_n \Delta E_m) \leq \varepsilon, \]
for all $n, m \geq N$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Prove that there exists a measurable set $E$ on $[0, 1]$ such that
\[ \lim_{n \to \infty} m(E_n \Delta E) = 0. \]

Problem 3. Evaluate the limit,
\[ \lim_{n \to \infty} \int_0^1 \frac{nx^ndx}{(1 + x^n)^2}, \]
and justify your answer.

Problem 4. Let us enumerate all rational points on $[0, 1]$,
\[ \mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}. \]
Consider the function
\[ f(x) = \sum_{n=1}^{\infty} \frac{|x - r_n|}{n^{3/2}} \]
on $[0,1]$. Prove that the function $f(x)$ is differentiable almost everywhere.

Problem 5. Prove that if $f_n$ converges to $f$ in $L^p(E)$ and $g_n$ converges to $g$ in $L^q(E)$, where
\[ \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad p > 0, \quad q > 0, \quad r \geq 1, \]
then $f_ng_n$ converges to $fg$ in $L^r(E)$.

Problem 6. Let
\[ f_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k^{2/3}}. \]
Prove that there exists an integrable function $f(x)$ on the interval $[0, 2\pi]$ such that
\[ \lim_{n \to \infty} \int_{[0,2\pi]} |f(x) - f_n(x)|dx = 0. \]
QUALIFYING EXAM IN REAL ANALYSIS
IUPUI, AUGUST 2013

PAVEL BLEHER

Problem 1. Let $E$ be a set on the line of a finite outer measure, $m^*E < \infty$, and
$$S = \sup_{\text{measurable } A, A \subset E} mA.$$ Prove that $S = m^*E$ if and only if $E$ is measurable.

Problem 2. Prove that if $f(x)$ is an absolutely continuous function on the segment $[0, 1]$ such that $f'(x) = 1$ almost everywhere, then $f(x) = x + C$, where $C$ is a constant.

Problem 3. Let $\{f_n(x), n = 1, 2, \ldots\}$ be a sequence of integrable functions on the segment $[0, 1]$ such that
$$\sum_{n=1}^{\infty} \|f_n\|_{L^1[0,1]} < \infty.$$ Prove that the series
$$\sum_{n=1}^{\infty} \sin f_n(x)$$ converges in $L^1[0,1]$.

Problem 4. Let $F(f)$ be a bounded linear functional on $L^1[0,1]$ such that $F(f) = 0$ for all $f$ such that
$$\int_0^1 f(x)dx = 0.$$ Prove that then there is a real number $C$ such that for all $f \in L^1[0,1],$
$$F(f) = C \int_0^1 f(x)dx.$$  

Problem 5. Prove that if $f_n$ converges to $f$ in $L^3[0,1]$, then $(f_n)^2$ converges to $f^2$ in $L^1[0,1]$.

Problem 6. Let $f$ be an integrable function on the segment $[0,1]$ such that for any $0 \leq a < b \leq 1,$
$$\left| \int_a^b f(x)dx \right| \leq (b - a)^2.$$ Prove that $f = 0$ almost everywhere.
Problem 1. Prove that if $A$ is a set on $[0, 1]$ and $A_n$ its $(1/n)$-neighborhood, i.e.,

$$A_n = \left\{ x \in \mathbb{R} : \exists y \in A, |x - y| < \frac{1}{n} \right\},$$

then

$$\bigcap_{n=1}^{\infty} A_n = \overline{A},$$

where $\overline{A}$ is the closure of $A$.

Problem 2. Prove that the collection of all measurable sets on $[0, 1]$ is separable, so that there exists a countable collection $\mathcal{M}_0 = \{E_1, E_2, \ldots\}$ of measurable sets on $[0, 1]$ such that for any measurable set $E$ on $[0, 1]$ and any $\varepsilon > 0$ there exists a set $E_n \in \mathcal{M}_0$ such that

$$m(E \Delta E_n) \leq \varepsilon,$$

where $E \Delta E_n = (E \setminus E_n) \cup (E_n \setminus E)$.

Problem 3. Let us enumerate all rational points on $[0, 1]$,

$$\mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}.$$ 

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2|x - r_n|^{1/2}}$$

on $[0, 1]$. Prove that the series defining the function $f(x)$ converges for almost all $x$ and $f$ is integrable.

Problem 4. Prove that if a sequence $\langle f_n \rangle$ of integrable functions on a measurable set $E$ converges to a function $f$ in such a way that

$$\int_E |f_n(x) - f(x)| \, dx \leq \frac{1}{2^n}, \quad n = 1, 2, \ldots,$$

then $f_n(x)$ converges to $f(x)$ a.e. on $E$.

Problem 5. Prove that if $f \in L^2[0, 1]$ and

$$\int_0^1 f(x) \, dx = 0,$$

then

$$\int_0^1 |f(x) - x|^2 \, dx \geq \frac{1}{4}.$$
Problem 6. Let $f(x)$ be an absolutely continuous function on an interval $[a, b]$ such that $f'(x) \in L^2[a, b]$. Prove that there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^{1/2}$$

for all $x, y \in [a, b]$. 
Problem 1. Let $f$ be a continuously differentiable function on $[0, 2\pi]$ such that $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$. Prove that for any $\varepsilon > 0$, there is a trigonometric polynomial,

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \sin(kx) + b_k \cos(kx)],$$

such that

$$\sup_{0 \leq x \leq 2\pi} |f(x) - g(x)| + \sup_{0 \leq x \leq 2\pi} |f'(x) - g'(x)| \leq \varepsilon.$$

Problem 2. Prove that for any Lebesgue measurable set $E$ on $[0, 1]$ and any $\varepsilon > 0$, there is a finite collection of disjoint intervals $(a_j, b_j), 1 \leq j \leq n$, on $[0, 1]$ such that all the end-points $\{a_j, b_j\}$ are rational and

$$m\left(E \Delta \bigcup_{j=1}^{n} (a_j, b_j)\right) \leq \varepsilon,$$

where $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Problem 3. Let $f$ satisfy a Lipschitz condition on $[0, 1]$, so that there is a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x$ and $y$ on $[0, 1]$. Prove that $f$ is differentiable almost everywhere on $[0, 1]$.

Problem 4. Let $f \in L^2[0, 1]$. Prove that

$$\inf_{-\infty < a < \infty} \int_{0}^{1} |f(x) - a|dx \leq \sqrt{\int_{0}^{1} f^2(x)dx - \left(\int_{0}^{1} f(x)dx\right)^2}.$$

Problem 5. The Sierpinski carpet is a set on the plane which is constructed as follows. We begin with a square. The square is cut into 9 congruent subsquares in a 3-by-3 grid, and the open central subsquare is removed. The same procedure is then applied recursively to the remaining 8 subsquares, ad infinitum. Prove that the planar Lebesgue measure of the Sierpinski carpet is equal to 0.

Problem 6. Prove that for any $\infty > p \geq 1$ there is a constant $C_p > 0$ such that for any integrable function $f$ on $[0, 1]$,

$$\|\ln(1 + |f|)\|_{L^p[0,1]} \leq C_p \left(1 + \|f\|_{L^1[0,1]}\right).$$
1. Assume that $B_n \subset A_n \subset \mathbb{R}$ for $n = 1, 2, \ldots$ are measurable sets. Prove that
$$m\left(\bigcup_{n=1}^{\infty} A_n \setminus \bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} (m(A_n) - m(B_n)).$$

2. Let $A_1, A_2, A_3, A_4$ be measurable subsets of $[0, 1]$, such that
$$\sum_{k=1}^{4} m(A_k) > 3.$$ Prove that
$$m\left(\bigcap_{k=1}^{4} A_k\right) > 0.$$ 

3. Consider a sequence of functions $f_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, convergent to $f$ in $L^1(\mathbb{R})$ and to $g$ in $L^2(\mathbb{R})$. Prove that $f = g$ a.e.

4. Let $E$ be a measurable subset of $\mathbb{R}$ and $f \in L^1(E)$. Set $A_n = \{x \in E : |f(x)| \geq n\}$ for $n = 1, 2, \ldots$. Prove that
$$\sum_{n=1}^{\infty} m(A_n) \leq \int_{E} |f|.$$ 

5. Let $p, q, r \in (1, \infty)$ with $1/p + 1/q + 1/r = 1$. Prove that for every functions $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ and $h \in L^r(\mathbb{R})$
$$\int_{\mathbb{R}} |fgh| \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r.$$ 

6. Let $f \in L^1([0, 1])$. Prove that $\sqrt{1 + f^2} \in L^1([0, 1])$ and
$$\sqrt{1 + (\|f\|_1)^2} \leq \left\|\sqrt{1 + f^2}\right\|_1.$$ 

Hint: For inequality, use Jensen’s inequality.
1. Prove that for every \( \varepsilon > 0 \) there exists an open set \( A \subset \mathbb{R} \), dense in \( \mathbb{R} \), of Lebesgue measure less than \( \varepsilon \).

2. Prove that if a function \( f : [0,1] \to \mathbb{R} \) has bounded variation and \( \inf\{|f(x)| : x \in [0,1]\} > 0 \), then \( 1/f \) also has bounded variation.

3. Let \( a < b \) be real numbers. Let \( f : [a, b] \to \mathbb{R} \) be an absolutely continuous function with bounded derivative and such that \( f(a) = 0 \). Prove that
\[
\sup\{|f(x)| : x \in [a, b]\} \leq \sqrt{(b - a) \int_{a}^{b} (f'(x))^2 \, dx}.
\]

4. Give example of a function \( f : [0,1] \to \mathbb{R} \) which is integrable, but is not essentially bounded on any subinterval of \([0,1]\).

5. Show that \( f \mapsto \sqrt{\|f\|_1 \cdot \|f\|_\infty} \) is not a norm in \( L^\infty([0,1]) \).

6. For \( n = 1, 2, 3, \ldots \) define a function \( f_n : [0,1) \to \mathbb{R} \) as follows. Let \( k(x,n) = \lfloor 2^n x \rfloor \) (the largest integer smaller than or equal to \( 2^n x \)); let
\[
f_n(x) = \begin{cases} 
0 & \text{if } k(x,n) \text{ is even}, \\
1 & \text{if } k(x,n) \text{ is odd}.
\end{cases}
\]

Does there exist a subsequence of the sequence \( (f_n)_{n=1}^{\infty} \) convergent almost everywhere? Explain.
Problem 1. Consider the space \( l \) of sequences \( x = (x_1, x_2, \ldots) \) such that the limit, \( \lim_{n \to \infty} x_n \), exists. Define the norm in \( l \) as the \( l^\infty \)-norm,
\[
\|x\|_\infty = \sup_n |x_n|.
\]
Is the space \( l \) complete or not? Explain.

Problem 2. Prove that the linear functional
\[
F[f] = \int_{0}^{1} xf(x) \, dx
\]
on \( L^3[0, 1] \) is bounded and evaluate its norm.

Problem 3. Let \( f \in L^1(-\infty, \infty) \). Find the limit,
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \left( \cos x + \frac{\sin^3 x}{2} \right)^n \, dx,
\]
and justify your answer.

Problem 4. Let \( C \) be the Cantor ternary set,
\[
C = [0, 1] \setminus \left[ \left( \frac{1}{3}, \frac{2}{3} \right) \cup \left( \frac{1}{9}, \frac{2}{9} \right) \cup \left( \frac{7}{9}, \frac{8}{9} \right) \cup \ldots \right],
\]
and let
\[
E = \{x + y \mid x \in C, \ y \in \mathbb{Q}\},
\]
where \( \mathbb{Q} \) is the set of rational numbers. Find the Lebesgue measure of \( E \) and justify your answer.

Problem 5. Prove that if \( f \in L^1(-\infty, \infty) \) and
\[
g(x) = \int_{-\infty}^{\infty} f(x - y)e^{-|y|} \, dy,
\]
then \( g \in L^1(-\infty, \infty) \) and
\[
\|g\|_1 \leq 2\|f\|_1.
\]

Problem 6. Let \((X, \mathcal{A}, \mu)\) be a measure space. Prove that if \( f \in L^6[X] \) and \( g \in L^3[X] \), then \( fg \in L^2[X] \).
Problem 1. The Riemann function \( R(x) \) on the segment \([0, 1]\) is defined as

\[
R(x) = \begin{cases} 
0, & \text{when } x \text{ is irrational or } x = 0, \\
\frac{1}{q}, & \text{when } x = \frac{p}{q} > 0, & \text{where } p \text{ and } q \text{ are relatively prime.}
\end{cases}
\]

Is there a sequence \( \langle f_n \rangle \) of continuous functions on \([0, 1]\) which converges to \( R(x) \) at every point \( x \in [0, 1] \)? Explain.

Problem 2. Let us enumerate all rational points on the segment \([0, 1]\),

\[ \mathbb{Q} \cap [0, 1] = \{ r_1, r_2, \ldots \}. \]

Define the function

\[
f(x) = \sum \frac{1}{2^n}.
\]

Prove that

1. the function \( f(x) \) is discontinuous at every rational point,
2. the function \( f(x) \) is continuous at every irrational point,
3. the function \( f(x) \) is differentiable almost everywhere,
4. \( f'(x) = 0 \) almost everywhere.

Problem 3. Let \( E_1 \subset E_2 \subset \ldots \) be an increasing sequence of Lebesgue measurable sets on the line, such that the set \( E = \bigcup_{n=1}^{\infty} E_n \) has a finite Lebesgue measure. Prove that for any set \( A \subset \mathbb{R} \),

\[
\lim_{n \to \infty} m^*(A \cap E_n) = m^*(A \cap E).
\]

Problem 4. Let \( f \in L^1(-\infty, \infty) \). Find the limit,

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{f(x)e^{nx}dx}{1 + e^{nx}},
\]

and justify your answer.

Problem 5. Consider the functional

\[
F(f) = \int_0^1 e^x f(x)dx.
\]

Prove that \( F \) is a linear bounded functional on \( L^4[0, 1] \) and calculate the norm of \( F \) on \( L^4[0, 1] \).
Problem 6. Prove that if \( f, g \in L^1(-\infty, \infty) \) and
\[
h(x) = \int_{-\infty}^{\infty} f(x - e^y)g(y)dy,
\]
then \( h \in L^1(-\infty, \infty) \) and
\[
\|h\|_1 \leq \|f\|_1 \|g\|_1.
\]
Problem 1. Does there exist a measurable set of non-zero measure which is closed and has no interior points?

Problem 2. Let \( \{f_n\} \) be a sequence of measurable functions on \((0, 1)\). Show that the set
\[
E = \{x \in (0, 1) | \{ f_n(x) \} \text{ is a convergent sequence} \}
\]
is measurable.

Problem 3. Enumerate all rational points on \([0, 1]\), so that
\[
\mathbb{Q} \cap [0, 1] = \{ r_1, r_2, \ldots \}.
\]
Define the function
\[
f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \theta(x - r_n), \quad 0 \leq x \leq 1,
\]
where
\[
\theta(x) = \begin{cases} 
1, & x \geq 0, \\
0, & x < 0.
\end{cases}
\]
Prove that there exists a point \( 0 < x_0 < 1 \) such that \( f(x) \) is differentiable at \( x_0 \).

Problem 4. Let \( g(x) \) be a continuous periodic function of period 1 on \( \mathbb{R} \). Prove that for any integrable function \( f(x) \) on \([0, 1]\),
\[
\lim_{n \to \infty} \int_{0}^{1} f(x) g(nx) \, dx = \int_{0}^{1} f(x) \, dx \int_{0}^{1} g(x) \, dx
\]

Problem 5. Prove that the operator
\[
A : f(x) \mapsto \int_{-\infty}^{+\infty} f(x-y)e^{-y^2} \, dy
\]
is a bounded operator in \( L^2(-\infty, \infty) \).

Problem 6. Let \((X, d)\) be a compact metric space. Let \( \{f_n\} \) be a sequence of continuous real valued functions on \( X \) such that \( f_n(x) \leq f_{n+1}(x) \) for all \( n, x \). Assume that the sequence \( \{f_n\} \) is pointwise convergent and let
\[
f(x) = \lim_{n \to \infty} f_n(x).
\]
Show that \( f(x) \) is continuous if and only if the sequence \( \{f_n\} \) converges uniformly.
Problem 1. Prove that the space $C^1[0, 1]$ of continuously differentiable functions on $[0, 1]$, with the norm,

$$
\|f(x)\|_{C^1[0, 1]} = \sup_{0 \leq x \leq 1} (|f(x)| + |f'(x)|),
$$

is separable.

Problem 2. Find a linear function $ax + b$ such that the norm,

$$
\|ax + b - e^x\|_{L^2[0, 1]},
$$

is minimal.

Problem 3. Let $f(x)$ be a continuous function on $\mathbb{R}$, and let $g(x)$ be a measurable function on $[0, 1]$. Is the function $f(g(x))$ measurable? Explain.

Problem 4. Is there an integrable function $f(x)$ on $[0, 1]$ such that for any $0 < a < b < 1$,

$$
\|f(x)\|_{L^2[a, b]} = \infty.
$$

Explain.

Problem 5. Is there an open connected set in $\mathbb{R}^2$ of finite Lebesgue measure, which has a non-empty intersection with any straight line? Explain.

Problem 6. Prove that there is a constant $C > 0$ such that for any continuous function $f$ on $[0, 1]$,

$$
\|\ln(1 + |f|)\|_{L^4[0, 1]} \leq C\|f\|_{L^2[0, 1]}.
$$
Problem 1. Prove that any set $X$ on the line is separable, i.e., there exists an at most countable set $A \subset X$ whose closure contains $X$.

Problem 2. Is there a measurable set $E$ on $[0, 1]$ such that for any $0 \leq a < b \leq 1$,
$$m(E \cap [a, b]) = \frac{b - a}{2},$$
where $m$ stands for the Lebesgue measure?

Problem 3. Prove that if $f$ is integrable on $[0, 1]$, then
$$\lim_{n \to \infty} n \int_0^1 \sin(x/n) f(x) \, dx = \int_0^1 x f(x) \, dx.$$

Problem 4. Prove that if functions $f_1$, $f_2$ are absolutely continuous on $[0, 1]$, then the function $f(x) = \max\{f_1(x), f_2(x)\}$ is absolutely continuous as well.

Problem 5. Let $f_1, f_2, \ldots$ be a sequence of nondecreasing absolutely continuous functions on the closed interval $[0, 1]$ such that the series
$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$
converges at every point $x \in [0, 1]$. Prove that the function $f$ is absolutely continuous.

*Hint.* Consider the series of the derivatives,
$$\sum_{n=1}^{\infty} f'_n(x),$$
and prove that it converges a.e. to an integrable function.

Problem 6. Suppose that $f \in L^3[0, 1]$. Prove that
$$\|xf(x)\|_2 \leq \|f(x)\|_3^{7/6}.$$

*Hint.* Use the Hölder inequality.
Problem 1. Suppose that \( A \subset \mathbb{R} \) is a measurable set. Prove that the set
\[
B = A \cup (-A), \quad -A \equiv \{x : -x \in A\},
\]
is measurable and
\[
mA \leq mB \leq 2mA,
\]
where \( m \) is the Lebesgue measure.

Problem 2. Prove that for any integrable function \( f \) on \([a, b]\),
\[
\lim_{n \to \infty} \int_{a}^{b} f(x) \sin^3(nx) \, dx = 0.
\]

Problem 3. Prove that there is no function \( f(x) \in L^{\infty}[0, 1] \) such that
\[
\lim_{n \to \infty} \int_{0}^{1} [f(x)]^n \, dx = 2.
\]

Problem 4. For a given \( \varepsilon > 0 \), construct an open dense set \( O \) on \([0, 1]\) such that \( mO < \varepsilon \), where \( m \) is the Lebesgue measure.

Problem 5. Prove that if \( f \in L^2[0, 1] \) and \( g \in L^3[0, 1] \) then \( fg \in L^{6/5}[0, 1] \).

Problem 6. Prove the existence of the limit,
\[
\lim_{n \to \infty} \int_{0}^{\infty} \frac{1 + x^n}{2 + x^n} e^{-x} \, dx,
\]
and find it.
Problem 1. Let $F$ be a closed subset of $[0,1]$ and $F_\varepsilon$, where $\varepsilon > 0$, its $\varepsilon$-neighborhood, i.e.,

$$F_\varepsilon = \{ x \in \mathbb{R} : \exists y \in F, |x - y| < \varepsilon \}.$$

Prove that

$$\lim_{\varepsilon \to 0} mF_\varepsilon = mF,$$

where $m$ is the Lebesgue measure. Does this statement hold in the case when $F$ is an arbitrary open subset of $[0,1]$? Explain.

Problem 2. Let $A$ and $B$ be two bounded closed subsets of $\mathbb{R}$. Prove that the set

$$A + B = \{ x + y : x \in A, y \in B \}.$$

is closed and

$$m(A + B) \geq mA + mB,$$

where $m$ is the Lebesgue measure.

Problem 3. Let $f_n : \mathbb{R} \to \mathbb{R}^+$ be a sequence of nonnegative integrable functions on the line such that

$$\lim_{n \to \infty} f_n(x) = 0$$

almost everywhere. Prove that

$$\lim_{n \to \infty} \int_{\mathbb{R}} \min_{1 \leq k \leq n} f_k(x) \, dx = 0.$$

Problem 4. Let $f$ be a nonnegative integrable function on the interval $[0,1]$ such that

$$\int_0^1 f(x) \, dx = 1,$$

and let $g(x) = \sqrt{f(1-x)}$. Prove that the function $g$ is integrable on $[0,1]$ and

$$\int_0^1 g(x) \, dx \leq 1.$$

Problem 5. Let $f$ be an integrable function on $[0,1]$ such that for any $0 \leq a < b \leq 1$,

$$\int_a^{(a+b)/2} f(x) \, dx = \int_{(a+b)/2}^b f(x) \, dx.$$

Prove that then the function $f$ is constant almost everywhere.
Problem 6. Let $f$ be a real valued function on the interval $(0, 1)$. A point $x$ is called a point of discontinuity of $f$ of the first kind, if the limits of $f$ at $x$ both from the right and from the left exist and they are not equal. Prove that the set of points of discontinuity of $f$ of the first kind is at most countable.

Hint. Prove first that the set of points of discontinuity of $f$ of the first kind, with the difference of the right and the left limits bigger then $1/n$, is at most countable for any $n = 1, 2, \ldots$. 
Problem 1. Suppose that $A \subset [0, 1]$ is a measurable set. Prove that the set

$$B = \cos A \equiv \{\cos x, \ x \in A\}$$

is measurable and $mB \leq 0.85mA$, where $m$ is the Lebesgue measure.

*Hint.* Use that $\sin 1 = 0.841 \ldots$

Problem 2. Prove that for any integrable function $f$ on $[a, b]$,

$$\lim_{n \to \infty} \int_a^b f(x) \sin^2(nx) dx = \frac{1}{2} \int_a^b f(x) dx.$$

Problem 3. Prove that there is no positive integrable function $f(x)$ on $[0, 1]$ such that

$$\lim_{n \to \infty} \int_0^1 [f(x)]^n dx = 2.$$

Problem 4. Let $A$ be a measurable set on $[0, 1]$. A point $x \in [0, 1]$ is called a *density point* of $A$ if

$$\lim_{\varepsilon \to 0} \frac{m(A \cap [x, x+\varepsilon])}{|\varepsilon|} = 1,$$

where $m$ stands for the Lebesgue measure. Prove that almost all points of the set $A$ are its density points.

*Hint.* Apply the theorem on the differentiation of an integral.

Problem 5. Let $f \in L^p(X)$, where $(X, \mathcal{A}, \mu)$ is a finite measure space, $\mu(X) < \infty$, and let $p > r \geq 1$. Prove that then $f \in L^r(X)$ and

$$\|f\|_r \leq (\mu X)^{\frac{1}{r} - \frac{1}{p}} \|f\|_p.$$

Problem 6. Prove that there exists the limit,

$$\lim_{n \to \infty} \int_0^1 \frac{e^{-x} \cos x}{nx^2 + \frac{1}{n}} dx,$$

and find it.

*Hint.* Make a change of variable.