

An Important Limit

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Objective

In this project we investigate an important limit which may not be intuitive.

Narrative

If you were asked to guess the value of a limit such as

$$\lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2} \quad (1)$$

before you knew the rules for computing such limits, you might easily have guessed the wrong number. You might have guessed that the value is 1, reasoning that when $x = -2$, both $x^2 - 4$ and $x + 2$ are 0, and that “Any number divided by itself is 1.”, so the limit should be 1. This guess is wrong, however, since

$$\lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2} = \lim_{x \rightarrow -2} \frac{(x - 2)(x + 2)}{x + 2} = \lim_{x \rightarrow -2} (x - 2) = -4.$$

(The flaw in your reasoning was that the rule is *not*, “Any number divided by itself is 1.”; the rule is “Any non-zero number divided by itself is 1.”!)

In this project we investigate

$$\lim_{h \rightarrow 0} (1 + h)^{1/h}.$$

It, like (1), might be nonintuitive. You might guess that this limit is 1 since, when $h = 0$, $1 + h = 1$ and, “1 to any power is 1”. This guess is wrong, however:

$$\lim_{h \rightarrow 0} (1 + h)^{1/h} = e = 2.71828182845905 \dots$$

Task

1. As a warm-up, we begin by investigating

$$\lim_{h \rightarrow 0} \frac{1 - \cos h}{h^2}.$$

(What would you guess this limit should be?)

- (a) Type the commands below into MATLAB in the order in which they are listed, ending each command line by hitting the Enter key. The effect of these commands is to create a table MPOS of values of $(1 - \cos h)/h^2$ for small positive values of h , and to display these values in a graph. Make sure you spell things correctly: remember that MATLAB is very sensitive to spelling (and misspelling). In particular, pay attention to the use of upper and lower case letters: MATLAB is case sensitive. (It distinguishes between “A” and “a”, for example.) Also be very careful with spaces and punctuation: MATLAB is sensitive to the use (and misuse) of these.

```
>> % Your name, today's date
>> % An Important Limit
>> clear all, close all
>> format long
>> % Task 1a: (1-cos(h))/h^2, h positive
>> for n=1:16 h = 0.75^n; MPOS(n,1) = h; MPOS(n,2) = (1-cos(h))/h^2; end
>> MPOS
>> figure(1)
>> plot(MPOS(:,1),MPOS(:,2),'+g:~')
>> axis([-1.0,1.0,0.0,0.75])
```

Before going on, it might be wise to check out the Figure 1 window.

In the above code we introduced the `plot` command: `plot(MPOS(:,1),MPOS(:,2),'+g:')` instructs MATLAB to plot the points whose coordinates are $(MPOS(n,1),MPOS(n,2))$, for all n ; the argument `'+g:'` specifies that the markers for each point should be +’s, the color should be green, and the markers should be connected with dotted lines (this is the “:” part of the argument). We also introduced the `axis` command: `axis([-1.0,1.0,0.0,0.75])` instructs MATLAB to limit plotting to values between -1.0 and 1.0 on the x -axis, and between 0.0 and 0.75 on the y -axis (so `xmin = -1.0, xmax = 1.0, ymin = 0.0, ymax = 0.75`).

- (b) Continue by typing the following commands into MATLAB in the order in which they are listed. The effect of these commands is to create a table `MNEG` of values of $(1 - \cos h)/h^2$ for small negative values of h , and to display `MNEG` and `MPOS` in a graph. (Remember that you can use the up- and down-arrows to repeat a line of code, and the left- and right-arrows to edit a line of code. These tricks can help cut down on the amount of typing you need to do.)

```
>> % Task 1b: (1-cos(h))/h^2, h negative
>> for n=1:16 h = -0.75^n; MNEG(n,1) = h; MNEG(n,2) = (1-cos(h))/h^2; end
>> MNEG
>> hold on
>> plot(MNEG(:,1),MNEG(:,2),'+r: ')
>> hold off
```

2. Repeat both parts of Task 1 for $\lim_{h \rightarrow 0}(1 + h)^{1/h}$. The only major changes that are necessary in the code for Task 1 involve:
- changing the commands `MPOS(n,2) = (1-cos h)/h^2` and `MNEG(n,2) = (1-cos h)/h^2` to the commands `MPOS(n,2) = (1+h)^(1/h)` and `MNEG(n,2) = (1+h)^(1/h)`, respectively,
 - changing `figure(1)` to `figure(2)`, and
 - changing the command `axis([-1.0,1.0,0.0,0.75])` to `axis([-0.5,0.5,0.0,5.0])`.

At this time, make a hard copy of MATLAB’s Command Window and hard copies of its Figure 1 and Figure 2 windows. If you made any typing errors, neatly draw a line through them and any resulting MATLAB output, by hand. Then:

3. Using a straightedge, very carefully draw the y -coordinate axis on the Figure 1 window hard copy. Then, again very carefully, transfer the y -tick marks and the y -tick mark labels to the y -coordinate axis (that you just drew). Finally, plot the point (along the y -axis) to which the points in Figure 1 are converging, and write the statement, “ $\lim_{h \rightarrow 0} \frac{1 - \cos h}{h^2} = \underline{\hspace{2cm}}$ ” filling in the blank with the correct limit.
4. Repeat all parts of Task 3 for the Figure 2 window and $\lim_{h \rightarrow 0}(1 + h)^{1/h}$ (now using a statement of the form, “ $\lim_{h \rightarrow 0}(1 + h)^{1/h} = \underline{\hspace{2cm}}$ ”).

Your lab report for this project will be your hard copies of the Command Window and the Figure 1 and Figure 2 windows.

Comments

The assignment $h = \pm 0.75^n$ controls the rate at which $h \rightarrow 0$: the rate could be increased by using $h = \pm 0.50^n$ instead of $h = \pm 0.75^n$, or increased even further by using $h = \pm 0.10^n$.