Solve 1!  Solve some!  Solve all!  
Be sure to tell us your reasoning and cite sources.  
Do it by March 8th and follow all of the rules on the website for submission!  
https://math.iupui.edu/math/highschoolmathcontest

Individual Problems

1. Label the ten vertices in the plane diagram to the right with digits 0 through 9, using each digit exactly once, so that (i) the values at the vertices of each of the three triangles add to a constant sum $T$; and (ii) the values at the vertices of each of the three pentagons add to another constant sum $P$. Two incorrect attempts are shown: one satisfies $T$, but not $P$; and the other satisfies $P$, but not $T$. Can you produce a labelling that satisfies both $T$ and $P$?

You may use the template available at https://math.iupui.edu/math/highschoolmathcontest

2. Let $P$ be a regular $n$-gon inscribed in the unit circle and let $Q$ be a regular $2n$-gon inscribed in the unit circle. Express the length of a side of $Q$ in terms of the length of a side of $P$.

3. Keep rolling a fair, six-faced die, and keep adding the outcomes until the running total exceeds 1000. When you stop, the running total must be 1001, 1002, ..., or 1006; but with what probabilities? (Approximate solutions are welcome if you cannot find an exact one.)

4. Let $x_0, ..., x_n$ be real numbers in the interval $[a, b]$, where both $a$ and $b$ are positive. Suppose that $x_0 + ... + x_n = 0$. Prove that $x_0^2 + x_1^2 + ... + x_n^2 \leq nab$.

5. Write 500 to 700 words (complete with references) on an application of mathematics to medicine.

Team Problems

1. Consider a polynomial $p_c(x) = x^2 + c$. The variable $c$ is thought of as a "parameter" which will be fixed at various chosen values throughout the problem. Let $p_n^c(x)$ denote the composition of $p_c(x)$ with itself $n$ times, for example $p_3^c(x) = p_c(p_c(p_c(x)))$. A point $x$ is called periodic for $p_c$ if there exists a natural number $n$ such that $p_n^c(x) = x$. The smallest such $n$ is called the "period" of $x$.

a) How many integer values of $x$ are periodic points for $p_1^c(x) = x^2 + 1$?

b) Can you find an integer value of the parameter $c$ such that $p_c(x)$ has an integer periodic point $x$ whose period is exactly 3?

c) Prove that for any integer parameter $c$ the polynomial $p_c(x)$ has at most finitely many integer points $x$ that are periodic.

d) Is there a uniform bound $M$ such that for all integer parameters $c$ the polynomial $p_c(x)$ has at most $M$ integer periodic points?

2. (a) $3^3 + 4^3 + 5^3 = 6^3$. Show how to cut a 6x6x6 cube into as few pieces as possible (composed of sets of contiguously connected 1x1x1 cubes) that can be reassembled into a 3x3x3 cube, a 4x4x4 cube, and a 5x5x5 cube.

(b) $10^3 + 9^3 = 12^3 + 1^3$. Same problem cutting a 12x12x12 cube into as few pieces as possible which, when combined with the 1x1x1 cube, can be reassembled into a 10x10x10 cube and a 9x9x9 cube.

You’re invited!

IUPUI cordially invites all participants, parents, and educators to the awards ceremony on April 12, 2019. Dr. Julia Arciero, Professor in the IUPUI Department of Mathematical Sciences, is giving the keynote speech. Winners will be announced and refreshments will be served. Details on our website and in separate e-mail to follow.
1. Label the ten vertices in the following plane diagram with digits 0 through 9, using each digit exactly once, so that
   (i) the values at the vertices of each of the three triangles add to a constant sum $T$; and
   (ii) the values at the vertices of each of the three pentagons add to another constant sum $P$.
You may use the template available at www.math.iupui.edu/2019-IUPUI-HS-math-contest

Below are two attempts: one satisfies $T$, but not $P$; and the other satisfies $P$, but not $T$. Can you produce a labelling that satisfies both $T$ and $P$?

(a) A $T$-constant labelling ($T=13$)  
(b) A $P$-constant labelling ($P=26$)
1. Let $P$ be a regular $n$-gon inscribed in the unit circle and let $Q$ be a regular $2n$-gon inscribed in the unit circle. Express the length of a side of $Q$ in terms of the length of a side of $P$.

2. Keep rolling a fair, six-faced die, and keep adding the outcomes until the running total exceeds 1000. When you stop, the running total must be 1001, 1002, ..., or 1006; but with what probabilities? (Approximate solutions are welcome if you cannot find an exact one.)

3. Let $x_1, ..., x_n$ be real numbers in the interval $[-a, b]$, where both $a$ and $b$ are positive. Suppose that $x_1 + ... + x_n = 0$. Prove that $x_1^2 + x_2^2 + ... + x_n^2 \leq nab$.

4. Write 500 to 700 words (complete with references) on an application of mathematics to medicine.

Team Problems

1. Consider a polynomial $p_c(x) = x^2 + c$. The variable $c$ is thought of as a “parameter” which will be fixed at various chosen values throughout the problem. Let $p_c^{on}(x)$ denote the composition of $p_c(x)$ with itself $n$ times, for example $p_c^3(x) = p_c(p_c(p_c(x)))$. A point $x$ is called periodic for $p_c$ if there exists a natural number $n$ such that $p_c^{on}(x) = x$. The smallest such $n$ is called the “period” of $x$.
   a) How many integer values of $x$ are periodic points for $p_{-1}(x) = x^2 - 1$?
   b) Can you find an integer value of the parameter $c$ such that $p_c(x)$ has an integer periodic point $x$ whose period is exactly 3?
   c) Prove for any integer parameter $c$ the polynomial $p_c(x)$ has at most finitely many integer points $x$ that are periodic.
   d) Is there a uniform bound $M$ such that for all parameters $c$ the polynomial $p_c(x)$ has at most $M$ periodic points?

2. (a) $3^3 + 4^3 + 5^3 = 6^3$. Show how to cut a 6x6x6 cube into as few pieces as possible (composed of sets of contiguously connected 1x1x1 cubes) that can be reassembled into a 3x3x3 cube, a 4x4x4 cube, and a 5x5x5 cube.
   (b) $10^3 + 9^3 = 12^3 + 1^3$. Same problem cutting a 12x12x12 cube into as few pieces as possible which, when combined with the 1x1x1 cube, can be reassembled into a 10x10x10 cube and a 9x9x9 cube.