IUPUI 2019 High School Math Contest Problems

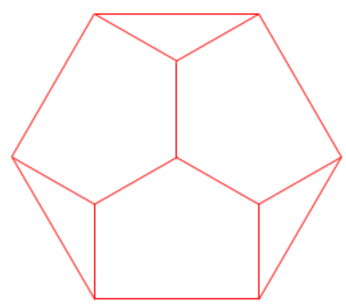
1. Label the ten vertices in the following plane diagram with digits 0 through 9, using each digit exactly once, so that

   (i) the values at the vertices of each of the three triangles add to a constant sum T; and

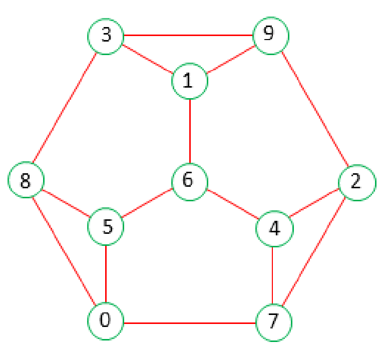
   (ii) the values at the vertices of each of the three pentagons add to another constant sum P.

You may use the template available at

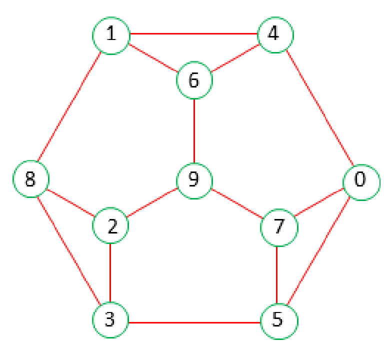
https://math.iupui.edu/math/highschoolmathcontest



Below are two attempts: one satisfies T, but not P; and the other satisfies P, but not T. Can you produce a labelling that satisfies both T and P?



(a) A T-constant labelling (T=13)



(b) a P-constant labelling (P=26)

2. Let P be a regular n-gon inscribed in the unit circle and let Q be a regular 2n-gon inscribed in the unit circle. Express the length of a side of Q in terms of the length of a side of P.

3. Keep rolling a fair, six-faced die, and keep adding the outcomes until the running total exceeds 1000. When you stop, the running total must be 1001, 1002, …, or 1006; but with what probabilities? (Approximate solutions are welcome if you cannot find an exact one.)
4. Let $x_1,...,x_n$ be real numbers in the interval [-a,b], where both a and b are positive. Suppose that $x_1+...+x_n = 0$. Prove that $x_1^2+x_2^2+....+x_n^2 <= nab$.

Team Problems

1. Consider a polynomial $p_c(x) = x^2 + c$. The variable c is thought of as a "parameter" which will be fixed at various chosen values throughout the problem. Let $p_c^{on}(x)$ denote the composition of $p_c(x)$ with itself n times, for example $p_c^{o3}(x) = p_c(p_c(p_c(x)))$. A point x is called periodic for p_c if there exists a natural number n such that $p_c^{on}(x) = x$. The smallest such n is called the "period" of x.
   a) How many integer values of x are periodic points for $p_{-1}(x) = x^2-1$?
   b) Can you find an integer value of the parameter c such that $p_c(x)$ has an integer periodic point x whose period is exactly 3?
   c) Prove for any integer parameter c the polynomial $p_c(x)$ has at most finitely many integer points x that are periodic.
   d) Is there a uniform bound M such that for all parameters c the polynomial $p_c(x)$ has at most M periodic points?

2. (a) $3^3 + 4^3 + 5^3 = 6^3$. Show how to cut a 6x6x6 cube into as few pieces as possible (composed of sets of contiguously connected 1x1x1 cubes) that can be reassembled into a 3x3x3 cube, a 4x4x4 cube, and a 5x5x5 cube.
   (b) $10^3 + 9^3 = 12^3 + 1^3$. Same problem cutting a 12x12x12 cube into as few pieces as possible which, when combined with the 1x1x1 cube, can be reassembled into a 10x10x10 cube and a 9x9x9 cube.