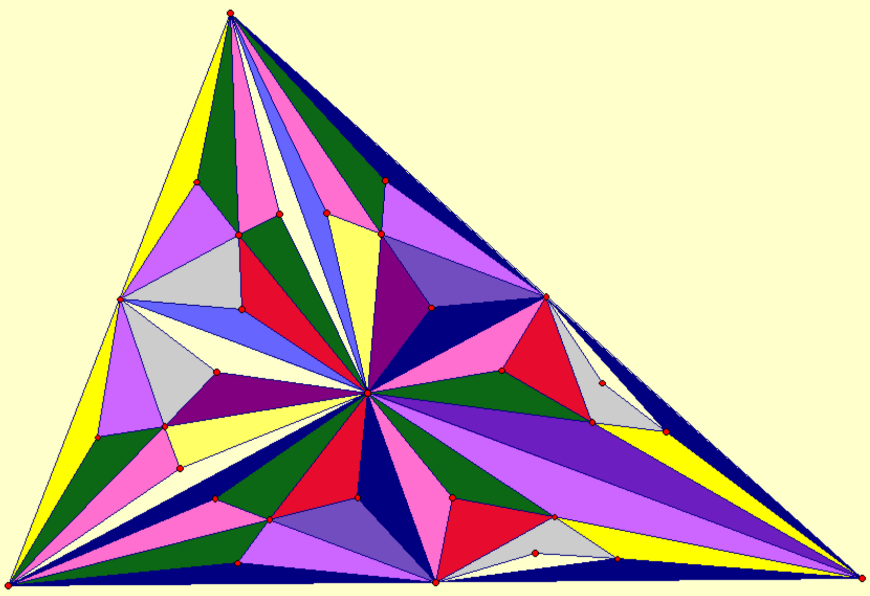
Mathematics of Chaos and Fractals



2011 IUPUI High School Mathematics Contest

Presented by
The IUPUI Department of Mathematical Sciences

STUDENT PRIZES

1 first place prize • $500
5 second place prizes • $200 each
10 third place prizes • $100 each

New This Year: A full four-year academic tuition scholarship will be awarded to the first place prizewinner. The winner must be directly admitted to the Purdue School of Science at IUPUI to major in any discipline and attend full-time.

Scholarships in the amount of $2,000 per year will be awarded to the remaining cash prizewinners who are directly admitted to the Purdue School of Science at IUPUI and attend full-time. These scholarships are renewable for four years, given satisfactory academic performance. Honorable mentions will receive a gift. All entrants will receive certificates honoring their participation.

MATHEMATICS DEPARTMENT AWARDS

Schools awarded the 1st place trophy in previous years:

2010 • Carmel High School
2009 • Park Tudor High School
2008 • Carmel High School
2007 • Hamilton Southeastern H.S.
2006 • Carmel High School
2005 • Carmel High School
2004 • Carmel High School
2003 • Hamilton Southeastern H.S.
2002 • Hamilton Southeastern H.S.
2001 • Ben Davis High School
2000 • Carmel High School
1999 • Roncalli High School
1998 • Brebeuf Jesuit Preparatory

CEREMONY

Prizewinners will be invited to an awards ceremony at IUPUI on Friday, May 13, 2011 from 4:00-6:30 p.m. Parents and teachers will also be invited. The program will feature an awards presentation, refreshments and a special talk by Dr. Rodrigo Perez, titled "Power Laws Rule."

ELIGIBILITY

This contest is open to students attending high school (grades 9-12) in the 15-county area of central Indiana: Bartholomew, Boone, Brown, Clinton, Hamilton, Hancock, Hendricks, Howard, Johnson, Madison, Marion, Morgan, Putnam, Shelby and Tipton.

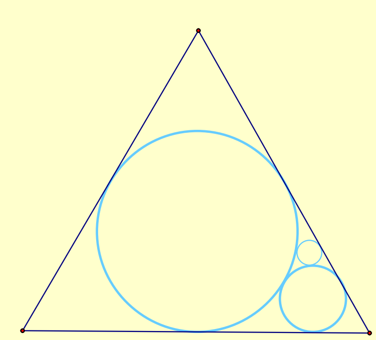
QUESTIONS

1. If $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ denotes the binomial coefficient, prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2$ is equal to 0 if n is an odd positive integer and $(-1)^{n/2}\binom{n}{n/2}$ if n is even and non-negative.

2. Show how a 13 by 13 square can be decomposed into 11 non-overlapping squares whose side lengths are integers (with sides parallel to the sides of the big square), if the largest subsquare is 7 by 7.

3. Prove or disprove: there exist infinitely many triples of positive integers (x, y, z) satisfying $x^2 + y^2 = z^2$ for which x and y are consecutive integers, as in the triples (3, 4, 5) and (21, 20, 29).

4. The equilateral triangle below has side length $2\sqrt{3}$. The three circles are tangent to the triangle and to each other as shown. Find the radii of the three circles.



5. Write an essay on the theme of the mathematics of chaos and fractals.

Contact Information:
www.math.iupui.edu/events/contest
IUPUI High School Mathematics Contest
Department of Mathematical Sciences
402 North Blackford Street, LD 270
Indianapolis, IN 46202-3216
(317) 274-MATH or contest@math.iupui.edu

ENTRIES: Mail your entry by Friday, April 15, 2011 to the address listed under Contact Information. You may obtain a copy of the questions, instructions for entering, and the cover page from your math teacher or the contest website: www.math.iupui.edu/events/contest. Solve the questions, giving your reasoning, not just the answers. Entries will be judged by professors in the IUPUI Department of Mathematical Sciences. Judging will be based on elegance of solution as well as correctness.

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2011 IUPUI HIGH SCHOOL MATH CONTEST ANSWERS

Solution to 1. Use the binomial theorem to expand both sides of the identity $(x+1)^n(x-1)^n = (x^2-1)^n$. This gives

$$\left(\sum_{k=0}^{n} \binom{n}{k} x^k\right)\left(\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k\right) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^{2k}.$$

Now look equate the coefficients of x^n on both sides:

$$\sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k}(-1)^{n-k} = 0 \text{ or } (-1)^{n/2}\binom{n}{n/2},$$

according as n is odd or even, because the right hand side is a polynomial in x^2. Now replace $\binom{n}{n-k}$ by $\binom{n}{k}$ and factor out $(-1)^n$ to obtain the desired identity:

$$\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k = 0 \text{ or } (-1)^{n/2}\binom{n}{n/2},$$

according as n is odd or even.

Solution to 2. One solution is pictured below in Figure 1.

Solution to 3. There are infinitely many triples (x, y, z) for which $x^2 + y^2 = z^2$ and x and y are consecutive positive integers. To prove this, write the Pythagorean triple (x, y, z) in terms of u and v:

$$x = u^2 - v^2, \quad y = 2uv, \quad z = u^2 + v^2,$$

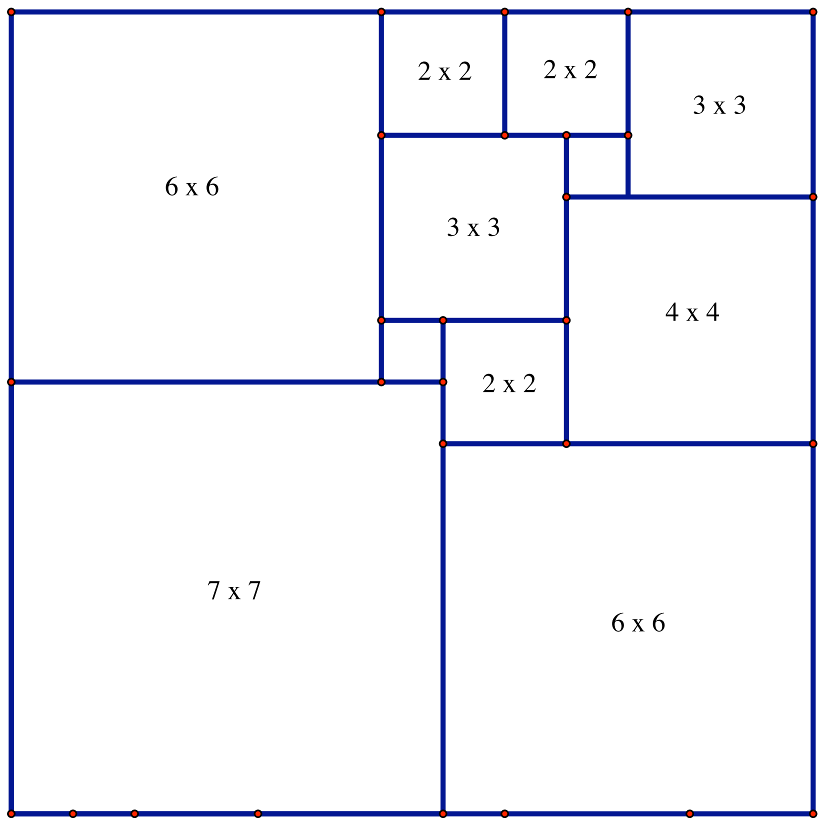

Figure 1: Solution to Problem 2.

and set $k = u^2 - v^2$ and $k - 1 = 2uv$. Subtracting the second equation from the first gives

$$1 = u^2 - v^2 - 2uv = (u - v)^2 - 2v^2.$$

We will first show there are infinitely many positive solutions of the equation $a^2 - 2b^2 = 1$. Note that $(a, b) = (3, 2)$ is one solution of $a^2 - 2b^2 = 1$. If (a, b) is any positive solution, then $(3a + 4b, 2a + 3b)$ is another solution, since

$$(3a + 4b)^2 - 2(2a + 3b)^2 = (9 - 8)a^2 + (24 - 24)ab + (16 - 18)b^2 = a^2 - 2b^2 = 1.$$

Now $(3a+4b, 2a+3b)$ is a larger solution than (a, b), so starting with $(a, b) = (3, 2)$, we successively get the new solutions $(a, b) = (17, 12), (99, 70), (577, 408)$ and so forth. This shows that the equation $a^2 - 2b^2 = 1$ has infinitely many solutions.

Now setting $v = b$ and $u = a + b$ gives infinitely many solutions of the displayed equation, so that $k = u^2 - v^2$ and $k - 1 = 2uv$ differ by 1, and $(x, y, z) = (k, k - 1, u^2 + v^2)$ is a solution of $x^2 + y^2 = z^2$. For example, if $(a, b) = (3, 2)$, then $(x, y, z) = (21, 20, 29)$; and if $(a, b) = (17, 12)$, then $(x, y, z) = (697, 696, 985)$. This completes the proof.

Note that the formulas for creating the new solution $(3a + 4b, 2a + 3b)$ from (a, b) can be found by multiplying out the expression $(a + b\sqrt{2})(3 + 2\sqrt{2}) = (3a + 4b) + (2a + 3b)\sqrt{2}$.

Solution to 4. Here is a solution using analytic geometry. See Figure 2 below.

Let the length of a side of the original equilateral triangle be $a = 2\sqrt{3}$. In the diagram the following lines are shown:

$$L_1: \quad y = -\sqrt{3}\left(x - \frac{a}{2}\right), \quad L_2: \quad y = \sqrt{3}\left(x + \frac{a}{2}\right)$$

$$L_3: \quad y = \sqrt{3}\left(x - \frac{a}{6}\right)$$

$$L_4: \quad y = 3\sqrt{3}\left(x - \frac{5a}{18}\right)$$

$$L_5: \quad y = \frac{1}{\sqrt{3}}\left(x - \frac{a}{2} + \frac{\sqrt{3}a}{12}\right) + \frac{a}{4}.$$

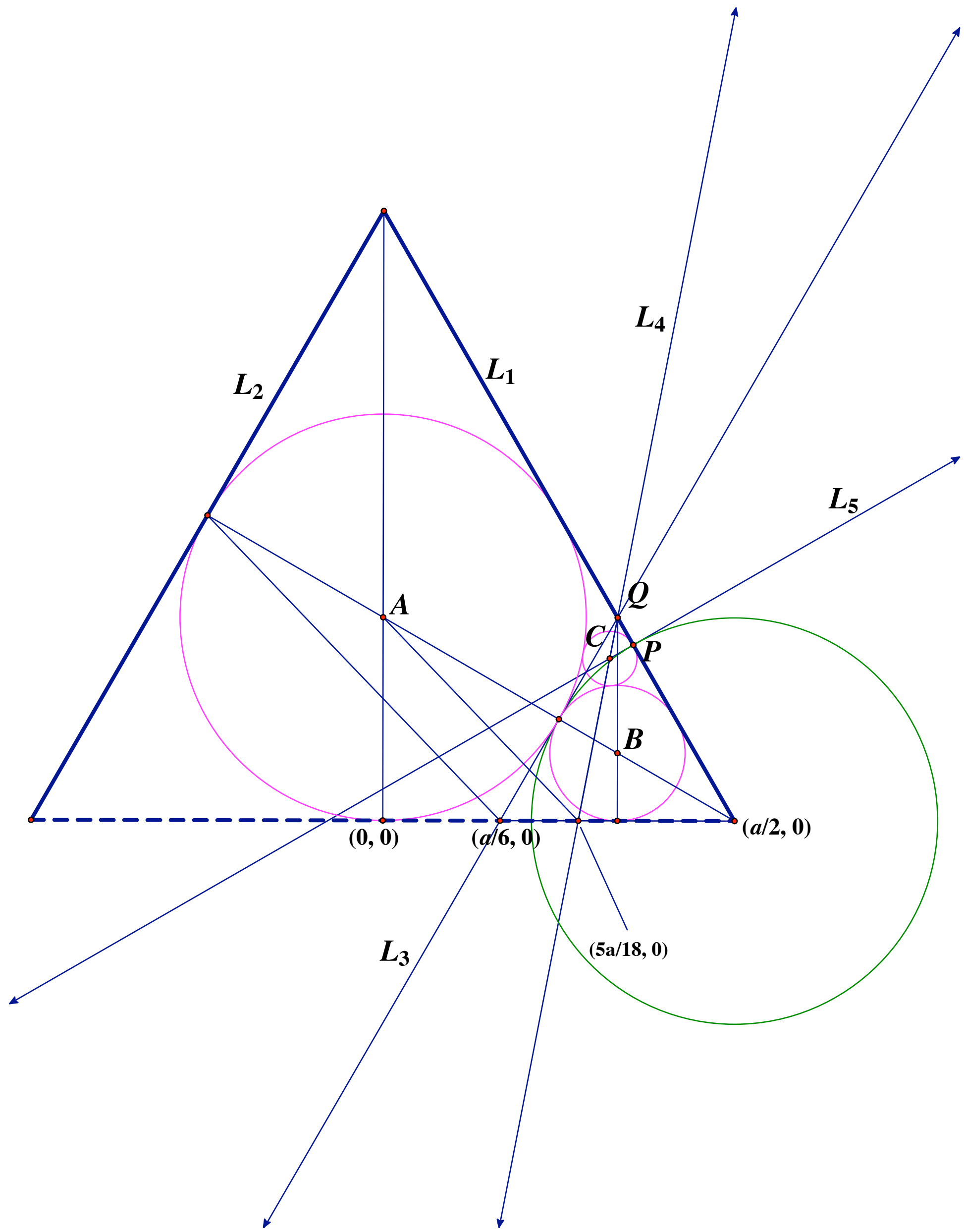

Figure 2: Solution to Problem 4.

The line L_4 connects the point $\left(\frac{5a}{18}, 0\right)$, which is 1/3 of the way from $\left(\frac{a}{6}, 0\right)$ to $\left(\frac{a}{2}, 0\right)$, to the point $Q = \left(\frac{a}{3}, \frac{\sqrt{3}a}{6}\right)$. The point $P = \left(\frac{a}{2} - \frac{\sqrt{3}a}{12}, \frac{a}{4}\right)$ is the point along the line L_1 which is $\sqrt{3}a/6$ units away from the point $\left(\frac{a}{2}, 0\right)$. The point $C = \left(\frac{a}{4} + \frac{\sqrt{3}a}{24}, \frac{3a}{18} - \frac{\sqrt{3}a}{12}\right)$ is the intersection of the lines L_4 and L_5, where L_5 is the perpendicular to L_1 through P.

The center A of the biggest circle has coordinates $\left(0, \frac{\sqrt{3}a}{6}\right)$. Its radius is $\frac{\sqrt{3}a}{6} = 1$.

The center B of the second largest circle is $\left(\frac{a}{3}, \frac{\sqrt{3}a}{18}\right)$, since B is on a vertical line through Q and therefore 1/3 of the way from $\left(\frac{a}{2}, 0\right)$ to the point A. The radius of circle B is $\frac{\sqrt{3}a}{18} = \frac{1}{3}$.

The radius of circle C is the distance from C to P, which is $a\left(\frac{\sqrt{3}}{6} - \frac{1}{4}\right) = 1 - \frac{\sqrt{3}}{2}$.

To complete the proof that these are the correct radii, we just have to show that the circles A, B, and C are tangent to each other. To do this we show that the distances between these points are just the sums of the respective radii. With some elementary algebra we get that

$$d(A,B) = \frac{2\sqrt{3}a}{9} = \frac{4}{3} = 1 + \frac{1}{3};$$
$$d(A,C) = a\left(\frac{\sqrt{3}}{3} - \frac{1}{4}\right) = 2 - \frac{\sqrt{3}}{2} = 1 + \left(1 - \frac{\sqrt{3}}{2}\right);$$
$$d(B,C) = a\left(\frac{2\sqrt{3}}{9} - \frac{1}{4}\right) = \frac{4}{3} - \frac{\sqrt{3}}{2} = \frac{1}{3} + \left(1 - \frac{\sqrt{3}}{2}\right).$$

This proves that the desired radii are indeed $1, \frac{1}{3}, 1 - \frac{\sqrt{3}}{2}$.

2011 IUPUI HIGH SCHOOL MATH CONTEST

First Prize Winner

Rebecca Chen, 11th Grade, Park Tudor. Teacher: Sarah Webster

Second Prize Winners

Youkow Homma, 11th Grade, Carmel High School. Teacher: Dan Perdun

Lyndon Ji, 11th Grade, Carmel High School. Teacher:

Nathaniel Smith, 10th Grade, Fishers High School. Teacher: John Drozd

Melinda Song, 10th Grade, Carmel High School. Teacher: Laura Diamente

Nina Sun, 10th Grade, Carmel High School. Teacher: Laura Diamente

Third Prize Winners

Mackenzie Baker-Robinson, 10th Grade, Fishers High School. Teacher: John Drozd

Steven Chen, 12th Grade, Carmel High School. Teacher: Vicki Tribul

Joseph Denton, 11th Grade, Hamilton Southeastern High School. Teacher: Letitia McCallister

William Etienne, 12th Grade, Avon High School. Teacher: Anthony Record

Victor Gonsalves, Fishers High School. Teacher: Louise Werner

Yushi Homma, 9th Grade, Carmel High School. Teacher: Dan Perdun

Jiefu Hou, 11th Grade, Carmel High School. Teacher: Jan Mitchener

Sloan Haywood, 11th Grade, Plainfield High School. Teacher: Brian Guth

Samuel Patterson, 10th Grade, Carmel High School. Teacher: Laura Diamente

Joel Whipple, 11th Grade, Avon High School. Teacher: Anthony Record

Paul Witcher, 11th Grade, Hamilton Southeastern High School. Teacher: Lisa Boyl

Weston Wright, 11th Grade, Avon High School. Teacher: Anthony Record

Honorable Mention Winners

Matthew Carlson, 11th Grade, Hamilton Southeastern High School. Teacher: Lisa Boyl

Max Clark, 12th Grade, Indiana University High School. Teacher: Steven Landy

Kristie Hsu, 11th Grade, Hamilton Southeastern High School. Teacher: Letitia McCallister

Abdulaziz Mohamed, 11th Grade, School of Knowledge. Teacher: Heba El-Shakmak

Usama Kamran, 10th Grade, Fishers High School. Teacher: Kathleen Robeson

Nusrat Khan, 12th Grade, Avon High School. Teacher: Anthony Record

Michael Masters, 10th Grade, Hamilton Southeastern High School. Teacher: Lisa Boyl

Sharmila Paul, Carmel High School. Teacher: Laura Diamente

Ryan Perry, 11th Grade, Hamilton Southeastern High School. Teacher: Bill Reed

BreeAnna Sawyer, 12th Grade, Avon High School. Teacher: Courtney Guth

Clayton Thomas, 9th Grade, Fishers High School. Teacher: Louise Werner

Phillip Witcher, 10th Grade, Hamilton Southeastern High School. Teacher: Lisa Boyl

School Award

Lindsey Low, 10th Grade, Brownsburg High School. Teacher: Micah Knobel