Math and Biology

About Sunflowers: The florets within the sunflower's cluster are arranged in a spiral pattern. Typically each floret is oriented toward the next by approximately the golden angle, 137.5°, producing a pattern of inter-connecting spirals where the number of left spirals and the number of right spirals are successive Fibonacci numbers. Typically, there are 34 spirals in one direction and 55 in the other; on a very large sunflower there could be 89 in one direction and 144 in the other. This pattern produces the most efficient packing of seeds within the flower head. (Source: http://en.wikipedia.org/wiki/Sunflower)

IUPUI 2010 High School Mathematics Contest

Presented by The IUPUI Department of Mathematical Sciences

STUDENT PRIZES:

*1 first prize $500
*5 second prizes $200 each
*10 third prizes $100 each

Scholarships in the amount of $2,500 per year will be awarded to cash prize winners who are directly admitted to the Purdue School of Science at IUPUI and attend full-time. This scholarship is renewable for four years, given satisfactory academic performance. Honorable mentions will receive a gift and all entrants will receive certificates honoring their participation.

MATHEMATICS DEPARTMENT AWARDS:
The 1st place team for 2009 was Park Tudor High School. Schools awarded the 1st place trophy in the past were:

- Carmel High School, 2008
- Hamilton Southeastern, 2007
- Carmel, 2006, 2005, 2004
- Hamilton Southeastern, 2003, 2002
- Ben Davis, 2001
- Carmel, 2000

CEREMONY:
Prize winners will be invited to an awards ceremony at IUPUI on Friday, May 14, 2010 from 4:00 to 6:30 p.m. Parents and teachers will also be invited. The program will feature an awards presentation, refreshments and a special talk by Dr. Robert Worth, titled "The Rhythmic Brain: Mathematics of Neurology."

ELIGIBILITY:
This contest is open to students attending high school (grades 9-12) in the 15-county area of central Indiana: Bartholomew, Boone, Brown, Clinton, Hamilton, Hancock, Hendricks, Howard, Johnson, Madison, Marion, Morgan, Putnam, Shelby and Tipton.

The photograph of the sunflower is the courtesy of Funny Kids T-Shirts: www.dizzy-ware.com.

QUESTIONS:

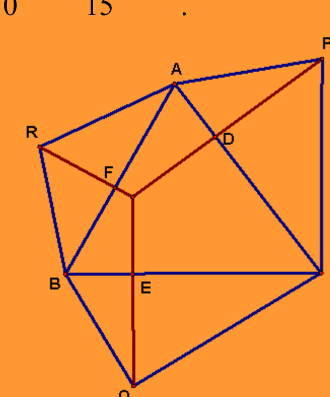
1. Show that in any pentagon (regular or not) there are two consecutive angles whose sum is at least 216º.

2. Show that the number $n^{n-1}-1$ is divisible by $(n-1)^2$ whenever $n \geq 2$.

3. Place the natural numbers in the grid as shown in the picture, filling every diagonal before continuing to the next diagonal. If the number 1 is assigned coordinates (1,1), the number 2 is assigned coordinates (1,2), the number 3 is assigned coordinates (2,1), etc., so that each number occupies a point with positive, integral coordinates in the first quadrant in the usual *x-y* plane, find the coordinates that are associated with the position of the number 2010.

.	.	.	.	.	.
11	.	.	.	.	.
7	12	.	.	.	.
4	8	13	.	.	.
2	5	9	14	.	.
1	3	6	10	15	.

4. Given triangle A, B, C and points P, Q, R outside of the triangle such that RA = AP, PC = CQ, QB = BR. let D be the foot of the perpendicular from P to AC, E the foot of perpendicular from Q to BC, F the foot of the perpendicular from R to AB. Show that the lines PD, QE, RF are concurrent, i.e., that they all meet in a single point.



5. Let U_n be the sequence of integers defined by the recursion formula $U_0 = 2$, $U_1 = 1$, $U_{n+1} = U_n + U_{n-1}$ for $n \geq 1$. Prove that:

$$\sum_{n=0}^{\infty} \frac{U_n}{10^{n+1}} = \frac{19}{89}$$

6. Write an essay of 500 to 700 words (complete with bibliography) on an application of mathematics to biology.

IUPUI
INDIANA UNIVERSITY–PURDUE UNIVERSITY INDIANAPOLIS

CONTACT INFORMATION:
www.math.iupui.edu/events/contest
IUPUI High School Mathematics Contest
Department of Mathematical Sciences
402 North Blackford Street, LD 270
Indianapolis, IN 46202-3216
(317) 274-MATH or contest@math.iupui.edu

ENTRIES:
Mail your entry by Friday, April 16, 2010 to the address listed under Contact Information. You may obtain a copy of the questions, instructions for entering, and the cover page from your math teacher or the contest website: www.math.iupui.edu/events/contest. Solve the questions, giving your reasoning, not just the answers. Entries will be judged by professors in the IUPUI Department of Mathematical Sciences. Judging will be based on elegance of solution as well as correctness.

2010 IUPUI HIGH SCHOOL MATH CONTEST ANSWERS

Solution to 1. If $\alpha_1, ..., \alpha_5$ are the angles in the pentagon, then adding the sums of two consecutive angles gives

$$(\alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3) + (\alpha_3 + \alpha_4) + (\alpha_4 + \alpha_5) + (\alpha_5 + \alpha_1) =$$

$$2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) = 2 \cdot (5 - 2) \cdot 180^\circ.$$

Therefore,

$$\frac{1}{5}\sum_{i=1}^{5}(\alpha_i + \alpha_{i+1}) = 6 \cdot \frac{180}{5} = 6 \cdot 36 = 216.$$

Since the average of the sum of consecutive angles is at least 216, it follows that at least one of the sums must be $\geq 216^\circ$. Thus, for some i, $\alpha_i + \alpha_{i+1} \geq 216^\circ$.

Solution to 2. Using the binomial theorem gives

$$n^{n-1} - 1 = (n - 1 + 1)^{n-1} - 1 = \sum_{k=0}^{n-1}\binom{n-1}{k}(n-1)^k - 1$$

$$= \sum_{k=1}^{n-1}\binom{n-1}{k}(n-1)^k$$

$$= (n-1)^2 + \binom{n-1}{2}(n-1)^2 + \binom{n-1}{3}(n-1)^3 + ...$$

is divisible by $(n-1)^2$ as long as $n-1 \geq 1$, i.e., $n \geq 2$ – so that the binomial coefficient $\binom{n-1}{1} = n-1$ is defined and nonzero.

Solution to 3. Consider the sequence $1, 2, 4, 7, 11, ...$ that forms the first column of numbers in the array, and call the n-th term of this sequence a_n, for $n \geq 1$. Consecutive differences of this sequence are

$$a_2 - a_1 = 1, a_3 - a_2 = 2, a_4 - a_3 = 3, ...$$

so for example, $a_4 = a_1 + (1+2+3) = 1 + (1+2+3)$. In general,

$$a_n = 1 + (1 + 2 + 3 + ... + n - 1) = 1 + \frac{(n-1)n}{2} = \frac{n^2 - n + 2}{2}.$$

Each term a_n is the first entry in the *diagonal* whose coordinates satisfy $x + y = n + 1$. We find the integer n for which

$$a_n < 2010 < a_{n+1}.$$

This will tell us that 2010 occurs in the n-th diagonal. We check that

$$a_{63} = \frac{63^2 - 63 + 2}{2} = 1954, \quad a_{64} = \frac{64^2 - 64 + 2}{2} = 2017;$$

hence, the number 2010 occurs in the 63-rd diagonal, six numbers from the bottom of that diagonal (since 2017 starts the next diagonal). SInce the coordinates of 2016 are (63, 1), backing up the diagonal by 6 steps give the coordinates (57, 7) for 2010. Hence the answer is (57, 7).

Solution to 4. Let a circle be drawn with center A through the points R and P, which is possible since $AR = AP$. SImilarly, draw a circle with center B through R and Q, and a circle with center C through Q and P. We need a lemma.

Lemma. If two circles A and B intersect in two points, R and R', then the segment RR' is perpendicular to the line AB joining the centers of the circles.

Proof. First show triangles ARB and $AR'B$ are congruent (SSS). If S is the intersection of RR' and AB, which exists since R and R' are obviously on opposite sides of the line AB, it follows that triangles ASR and ASR' are congruent (SAS). Hence the angles of the same names are congruent, implying they are right angles.

Now, if the circles with centers A and B intersect in two points in our problem, it follows from the lemma that the line through R perpendicular to side AB, which is the line joining the centers of the two circles, goes through the second point of intersection, R'. If circles A and B only intersect in one point R, then the perpendicular to AB through R is the common tangent line to circles A and B. In either case the line through R and F is the *radical axis* of circles A and B. Similarly, line PD is the radical axis of circles A and C, and line QE is the radical axis of circles B and C. We now use the theorem that the radical axes of three circles, taken two at a time, all intersect in one point. Hence, lines, PD, QE, and RF are concurrent.

Solution to 5. We compute the following:

$$(10^2 - 10 - 1)\sum_{n=0}^{\infty}\frac{U_n}{10^{n+1}} = \sum_{n=0}^{\infty}\frac{U_n}{10^{n-1}} - \sum_{n=0}^{\infty}\frac{U_n}{10^n} - \sum_{n=0}^{\infty}\frac{U_n}{10^{n+1}}$$

$$= 10U_0 + U_1 + \sum_{n=1}^{\infty}\frac{U_{n+1}}{10^n} - U_0 - \sum_{n=1}^{\infty}\frac{U_n}{10^n} - \sum_{n=1}^{\infty}\frac{U_{n-1}}{10^n}$$

$$= 20 + 1 - 2 + \sum_{n=1}^{\infty}\frac{U_{n+1} - U_n - U_{n-1}}{10^n} = 19 + 0 = 19,$$

because the recuurence $U_{n+1} - U_n - U_{n-1} = 0$ implies that the infinite sum in the last line is 0. If the infinite series in the problem is denoted by S, then this calculation shows that $89S = 19$, so $S = 19/89$.

2010 IUPUI HIGH SCHOOL MATH CONTEST

First Prize Winner

Rebecca Chen, 10th Grade, Park Tudor. Teacher: Tom Page

Second Prize Winners

Youkow Homma, 10th Grade, Carmel High School. Teacher: Kathie Freed

Lyndon Ji, 10th Grade, Carmel High School. Teacher: Kathie Freed

Michael Luo, 11th Grade, Carmel High School. Teacher: Kathie Freed

Shawn Qian, 12th Grade, Carmel High School. Teacher: Kathie Freed

Melinda Song, 9th Grade, Carmel High School. Teacher: Laura Diamente

Third Prize Winners

Roshni Bag, 9th Grade, Carmel High School. Teacher: Laura Diamente

Steven Chen, 11th Grade, Carmel High School. Teacher: Kathie Freed

Peter Ciaccia, 10th Grade, Brebeuf Jesuit Preparatory School. Teacher: Tim Kelaghan

Alan Gross, 6th Grade, Brownsburg High School. Teacher: Douglas Johnson

Iraj Hassan, 12th Grade, North Central High School. Teacher: Rick Shadiow

David Liang, 8th Grade, Carmel High School. Teacher: Janice Mitchener

Terry Ming, 10th Grade, Carmel High School. Teacher: Kathie Freed

Kevin Song, 11th Grade, Carmel High School. Teacher: Kathie Freed

Erin Van Wesenbeeck, 11th Grade, Fishers High School. Teacher: John Drozd

Raymond Watkin, 12th Grade, Franklin Community High School. Teacher: Timothy Kasper

Bosi Zhang, 11th Grade, Hamilton Southeastern High School. Teacher: Letitia McCallister

Tom Zhang, 11th Grade, Hamilton Southeastern High School. Teacher: Letitia McCallister

Honorable Mention Winners

Jacob Burton-Edwards, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Daniel Griffin, 10th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Aharon Hannan, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Kaitlyn Hartley, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Katharine Heinz, 11th Grade, Center Grove High School. Teacher: Karen Fruits

Avinash Inabathula, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Kayla Jansen, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Kalp Juthani, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Kelsey McBarron, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Di Mo, 12th Grade, Fishers High School. Teacher: John Drozd

Emily Mudd, 12th Grade, Hamilton Southeastern High School. Teacher: Letitia McCallister

Alexis Newton, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Joshua Oriez, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Samuel Patterson, 9th Grade, Carmel High School. Teacher: Laura Diamente

Ryan Roby, 12th Grade, North Central High School. Teacher: Sally Ernstberger

Kyle Ruschhaupt, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Tyra Salisbury, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Ryan Straut, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Anthony Vo, 11th Grade, Hamilton Southeastern High School. Teacher: Susan Wong

Michael Yeh, 11th Grade, Taylor High School. Teacher: Phil Spitler