Game Theory

象棋

Xiangqi

Xiangqi is known in the west as Chinese Chess. Each side has a royal piece known as the general. The object, as in western chess, is to checkmate him. Each side begins with two advisors, two elephants, two horses, two cannons and five soldiers, each with specialized moves.

Elephant

IUPUI 2009 High School Mathematics Contest

Presented by The IUPUI Department of Mathematical Sciences

Elephant

STUDENT PRIZES:

*1 first prize $500
*5 second prizes $200 each
*10 third prizes $100 each

Scholarships in the amount of $2,500 per year will be awarded to cash prize winners who are directly admitted to the Purdue School of Science at IUPUI and attend full-time. This scholarship is renewable for four years, given satisfactory academic performance Honorable mentions will receive a gift and all entrants will receive certificates honoring their participation.

MATHEMATICS DEPARTMENT AWARDS:

The 1st place team for 2008 was Carmel High School. Schools awarded the 1st place trophy in the past were:

- Hamilton Southeastern, 2007
- Carmel, 2006, 2005, 2004
- Hamilton Southeastern, 2003, 2002
- Ben Davis, 2001
- Carmel, 2000
- Roncalli, 1999
- Brebeuf Jesuit, 1998

CEREMONY:

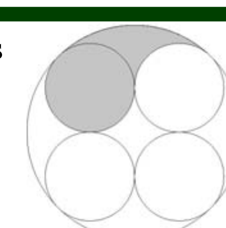
Prize winners will be invited to an awards ceremony at IUPUI on Friday, May 15, 2009 from 4:00 to 6:30 p.m. Parents and teachers will also be invited. The program will feature an awards presentation, refreshments and a special talk by economist Subir Chakrabarti, titled "Thinking Strategically: The Role of Game Theory."

ELIGIBILITY:

This contest is open to students attending high school (grades 9-12) in the 15-county area of central Indiana: Bartholomew, Boone, Brown, Clinton, Hamilton, Hancock, Hendricks, Howard, Johnson, Madison, Marion, Morgan, Putnam, Shelby and Tipton.

QUESTIONS:

1. In the figure to the right, all touching circles are tangent, and the radius of the four small circles is 1. What is the area of the shaded region?



2. In a triangle, the length of one side a is equal to one third of the sum of the lengths of the other two sides b and c. Show that the angle opposite side a is the smallest.

3. Find all the prime numbers p with the property that $8p^4-123$ is also a prime.

4. Each of three cards has an integer written on it. The three integers p, q, r satisfy the condition $0 \leq p < q < r$. Three players A, B, C mix the cards and pick one each. The number on the card they select is added to their scores. This process is repeated at most ten times, after which A has 20 points, B has 10 points, and C has 9 points. Also we know that B got the r card in the last round. Who received the q card in the first round?

5. Write an essay of 500 to 700 words (complete with bibliography) on an application of Game Theory.

ENTRIES:

Mail your entry by Friday, April 17, 2009 to the address below. You may obtain a copy of the questions, instructions for entering, and the cover page from your math teacher or the contest website. Solve the questions, giving your reasoning, not just the answers. Entries will be judged by professors in the IUPUI Department of Mathematical Sciences. Judging will be based on elegance of solution as well as correctness.

CONTACT INFORMATION: www.math.iupui.edu/news/contest
IUPUI High School Mathematics Contest
Department of Mathematical Sciences
402 North Blackford Street, LD 270
Indianapolis, IN 46202-3216
(317) 274-MATH or contest@math.iupui.edu

IUPUI
INDIANA UNIVERSITY–PURDUE UNIVERSITY INDIANAPOLIS

The photograph of Chinese chess pieces is courtesy of Wee Sen Goh, Senior Assistant Director (Design & Media) at Nanyang Technological University in Singapore.

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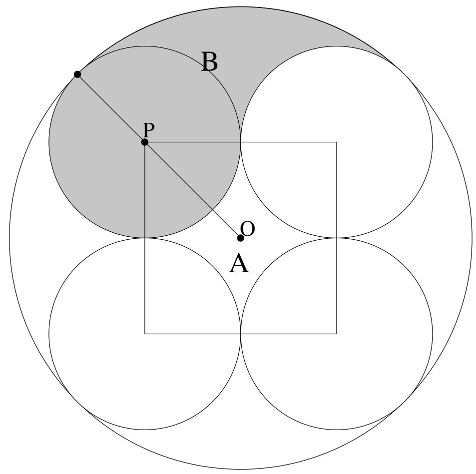
Answers to the 2009 IUPUI High School Mathematics Contest

1. In the figure, all touching circles are tangent, and the radius of the four small circles is 1. What is the area of the shaded region?



Solution. Let C be area of the large circle, B be the area of the region we seek, and A be the area of the 4-pointed star in the center. We have $C = 4B + A$.

Draw the square whose corners are the centers of the small circles. Its sidelength is 2 so its area is 4. So A equals 4 minus the area of a small circle (split into 4 segments), or $A = 4 - \pi$.

The distance $OP = \sqrt{2}$ so the radius of the large circle is $1 + \sqrt{2}$ and $C = \pi(1 + \sqrt{2})^2$.

Substituting and solving for B we find $B = \pi(1 + \sqrt{2}/2) - 1$.

2. In a triangle, the length of one side a is equal to one third of the sum of the lengths of the other two sides b and c. Show that the angle opposite side a is the smallest.

Solution. First we show side a is the shortest. By the triangle inequality, $c < a + b$. Substituting into $a = \frac{1}{3}b + \frac{1}{3}c$, $a < \frac{1}{3}b + \frac{1}{3}(a + b)$, which simplifies to $a < b$. Similarly, from $b < a + c$ we find $a < c$. Since a is the shortest side, it is opposite to the smallest angle.

3. Find all the prime numbers p with the property that $8p^4 - 123$ is also a prime.

Solution. We can check that for the primes $p = 2$ and $p = 5$, $8p^4 - 123$ is also prime. These are the only two primes for which this is true. To see this, note that the units digit of all other primes is 1, 3, 7, or 9. Because $1^4 = 1$, $3^4 = 81$, $7^4 = 2401$ and $9^4 = 6561$, the last digit of p^4 for any prime p other than 2 and 5 is a 1. Multiply by 8, and the last digit is an 8. Subtract 123 and the last digit is a 5. This means it is divisible by 5 and hence is not prime.

4. Each of three cards has an integer written on it. The three integers p, q, r satisfy the condition $0 \le p < q < r$. Three players A, B, C mix the cards and pick one each. The number on the card they select is added to their scores. This process is repeated at most ten times, after which A has 20 points, B has 10 points, and C has 9 points. Also, we know that B got the r card in the last round. Who received the q card in the first round?

Solution. Because a constant number of points, $p + q + r$, is awarded in each round, $p + q + r$ must divide the total number of points, 39. It follows that the number of rounds must be 1, 3, 13 or 39. As there were at most 10 rounds, there must have been 1 or 3. There can't have been only one round because B would have gotten the r card in that round so that $B = 10$, whereas A got 20 points, which is larger. Since this is not possible, there were exactly three rounds, and $p+q+r = 13$.

Observe that no player got all three of p, q and r since their total is 13. Thus each player has a duplication. Note also that $r \le 10$ because B got a total of 10, and $r \ge 7$, since the total for A cannot reach 20 if $r < 7$.

We have the following possibilities to check:

p	q	r
0	3	10
1	2	10
0	4	9
1	3	9
0	5	8
1	4	8
2	3	8
0	6	7
1	5	7
2	4	7

The only triples (p, q, r) that allow A to get 20 points are (10,3,0) and (8,4,1).

In the first case, A must have drawn 10,10,0, B must have drawn 0,0,10, and C 3,3,3.

In the second case, A must have drawn 8,8,4, B must have drawn 1,1,8, and C 4,4,1.

In each of our two solutions, C drew the q card on the first round.

2009 IUPUI HIGH SCHOOL MATH CONTEST

First Prize

Rebecca Chen, 9th Grade, Park Tudor. Teacher: Joseph Chamberlin

Second Prizes

Mohammad Aref, 11th Grade, School of Knowledge. Teacher: Heba Shakmak

Lyndon Ji, 9th Grade, Carmel High School. Teacher: Kathie Freed

Erika McGuire, 12th Grade, Warren Central High School. Teacher: Steven Landy

Shawn Qian, 11th Grade, Carmel High School. Teacher: Matthew Wernke

April Wang, 11th Grade, Park Tudor. Teacher: Joanne Black

Third Prizes

Justin Ahmann, 9th Grade, Zionsville Community High School. Teacher: Linda Gregg

Salman Alsaeede, 10th Grade, School of Knowledge. Teacher: Heba Shakmak

Steven Chen, 10th Grade, Carmel High School. Teacher: Janice Mitchener

Peter Ciaccia, 9th Grade, Brebeuf Jesuit. Teacher: Tim Kelaghan

Michael Luo, 10th Grade, Carmel High School. Teacher: Kathie Freed

Ryan Roby, 11th Grade, North Central High School. Teacher: Rick Shadiow

Mohamad Saltagi, 11th Grade, School of Knowledge. Teacher: Heba Shakmak

Jimmy Sun, 11th Grade, Carmel High School. Teacher: Kathie Freed

Jared Timmer, 11th Grade, Hamilton Southeastern. Teacher: Susan Wong

Tom Zhang, 10th Grade, Hamilton Southeastern. Teacher: Susan Wong

Honorable Mention Winners

Matthew Blandford, 10th Grade, Roncalli. Teacher: Sister Anne Frederick

Scott Blankenbaker, 9th Grade, Carmel High School. Teacher: Jan Mitchener

Kelly Commons, 12th Grade, Broad Ripple High School. Teacher: Peggy Boulden

Paul Glennan, 11th Grade, North Central High School. Teacher: Sally Ernstberger

William Gross, 11th Grade, Brownsburg High School. Teacher: Dan Schermer

Alan Gross, 5th Grade, Brownsburg High School. Teacher: Doug Johnson

Iraj Hassan, 11th Grade, North Central High School. Teacher: Jan Wendt

Katharine Heinz, 10th Grade, Center Grove High School. Teacher: Marcene Hensley

Youkow Homma, 9th Grade, Carmel High School. Teacher: Kathie Freed

Yiran Jiang, 11th Grade, Carmel High School. Teacher: Jan Mitchener

Christopher May, 12th Grade, Warren Central High School. Teacher: Steven Landy

Terry Ming, 9th Grade, Carmel High School. Teacher: Kathie Freed

Sawyer Morgan, 10th Grade, Hamilton Southeastern. Teacher: Susan Wong

Richard Ni, 10th Grade, Park Tudor. Teacher: Joanne Black

Samuel Smith, 10th Grade, Fishers High School. Teacher: Kathleen Robeson

Zachary Snider, 11th Grade, Hamilton Southeastern. Teacher: Susan Wong

Madeline Snipes, 8th Grade, Fishers High School. Teacher: Kathleen Robeson

Saya Wai, 12th Grade, Fishers High School. Teacher: John Drozd

Raymond Watkin, 11th Grade, Franklin Community. Teacher: Timothy Kasper

Michael Yeh, 10th Grade, Taylor High School. Teacher: Phil Spitler

Bosi Zhang, 10th Grade, Hamilton Southeastern. Teacher: Susan Wong