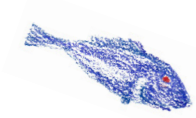
Statisticians have the job of trying to draw conclusions from a stream of data. The green fish and the blue fish have different tendencies to school or disperse. The tendency of the green (or blue) fish to school can be measured by the sample correlation matrix.

$$\begin{pmatrix} \mathrm{var}(x) & \mathrm{cov}(x,y) & \mathrm{cov}(x,z) \\ \mathrm{cov}(x,y) & \mathrm{var}(y) & \mathrm{cov}(y,z) \\ \mathrm{cov}(x,z) & \mathrm{cov}(y,z) & \mathrm{var}(z) \end{pmatrix}$$

IUPUI 2007 High School Mathematics Contest

Presented by The IUPUI Department of Mathematical Sciences



STUDENT PRIZES:

Scholarships in the amount of $2,500 per year will be awarded to 16 winners who are directly admitted to the Purdue School of Science at IUPUI and attend full-time. This scholarship is renewable for four years, subject to certain requirements.

Thirty top entrants will receive an interesting book on mathematics.

MATHEMATICS DEPARTMENT AWARDS:

The 1st place team for 2006 was Carmel High School. Schools awarded the 1st place trophy in the past were:

- Carmel, 2005
- Carmel, 2004
- Hamilton Southeastern, 2003
- Hamilton Southeastern, 2002
- Ben Davis, 2001
- Carmel, 2000
- Brebeuf Jesuit, 1998
- Roncalli, 1999

CEREMONY:

Prizewinners will be invited to an awards ceremony at IUPUI on Friday, May 11, 2007 from 4:00 to 6:30 p.m. Parents and teachers will also be invited. The program will feature an awards presentation, refreshments and a special talk on "A Data-scientist in Action."

ELIGIBILITY:

This contest is open to students attending high school (grades 9-12) in the fifteen-county area of central Indiana: Bartholomew, Boone, Brown, Clinton, Hamilton, Hancock, Hendricks, Howard, Johnson, Madison, Marion, Morgan, Putnam, Shelby and Tipton.

QUESTIONS:

1. Call a point in the plane "integer" if both of its coordinates are integers. Let a "configuration" be any random selection of 5 integer points. Prove that in any configuration, the line segment joining 2 of the points passes through another integer point (in some cases another point of the configuration).

2. The shortest side of a triangle has length 1 and the tangents of all of its angles are integers. Find the possible lengths of the other two sides.

3. Find the last two digits of the natural number $13^{12^{11^{\cdot^{\cdot^{\cdot^{3^2}}}}}}$.

4. The squares of an n by n "chessboard" are randomly assigned the numbers 1 through n^2. You are told the sum of every two squares that are adjacent, either vertically or horizontally (not diagonally).

   (a) In the case of a normal 8 by 8 chessboard, is this information always enough to determine which number is in each square?

   (b) What about a superlarge 2007 by 2007 chessboard?

5. Write an essay of 500 to 700 words (complete with bibliography) on an application of statistics to decision making.

ENTRIES:

Mail your entry by Friday, April 13, 2007 to the address below. You may obtain a copy of the questions, instructions for entering, and the cover page from your math teacher or the contest website. Solve the questions, giving your reasoning, not just the answers. Entries will be judged by professors in the IUPUI Department of Mathematical Sciences. Judging will be based on elegance of solution as well as correctness.

CONTACT INFORMATION: **www.math.iupui.edu/news/contest**

IUPUI High School Mathematics Contest
Department of Mathematical Sciences
402 North Blackford Street, LD 270
Indianapolis, IN 46202-3216
(317) 274-MATH or contest@math.iupui.edu

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for the original artwork.

INDIANA UNIVERSITY
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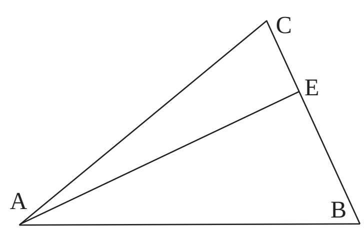
Answers to the 2007 IUPUI High School Mathematics Contest

1. Any integer point is of one of the four types (even, even), (even, odd), (odd, even) or (odd, odd). Any configuration of five points will duplicate one of these. So take two of the points such that x_1 and x_2 are both even or both odd, and y_1 and y_2 are both even or both odd. Then $x_1 + x_2$ and $y_1 + y_2$ are even, so $\frac{x_1+x_2}{2}$ and $\frac{y_1+y_2}{2}$ are integers. So the midpoint of the two points (x_1, y_1) and (x_2, y_2) is an integer point.

2. First note that the triangle is not a right triangle because the angles A, B and C all have tangents. Let A be the smallest angle. It satisfies $A \leq 60°$, so $\tan A \leq \sqrt{3}$. Since $\tan A$ is an integer it must be 1. Now using the angle sum formula for tangents twice, beginning with $A+B+C = 180°$, we obtain the identity

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

for any non-right triangle. This means $(\tan B, \tan C)$ is a solution of $1+x+y = xy$. The only integer points on this hyperbola are (2,3), (3,2), (-1,0) and (0,-1). From this we conclude that there is only one such triangle and the tangents of its angles are 1, 2 and 3.



The side of length 1 will be BC. Let AE be the altitude to BC. We know $\tan(B) = \frac{AE}{BE} = 2$ and $\tan(C) = \frac{AE}{CE} = 3$. These, together with $BE + CE = 1$ allow us to compute $AE = \frac{6}{5}$, $BE = \frac{3}{5}$ and $CE = \frac{2}{5}$. Now using the Pythagorean theorem with triangles ABE and ACE we find $AC = \frac{2\sqrt{10}}{5}$ and $AB = \frac{3\sqrt{5}}{5}$.

3. The last 2 digits of 13^n are the remainder after 13 is divided by 100. This remainder is written $13^n \bmod 100$. For example, $13^3 = 2197 = 21 \cdot 100 + 97$, so $13^3 = 97 \bmod 100$. In general, if we are working mod k the remainder will be in the range $0 \ldots k-1$.

Lemma: To find a product $ab \bmod k$ we may reduce a and b mod k first: $ab \bmod k = (a \bmod k)(b \bmod k) \bmod k$.

Proof: If $a = tk + r$ and $b = uk + s$, then $ab = (tuk + ru + st)k + rs$, so rs and ab will have the same remainder after division by k.

For example, $13^3 = 169 \cdot 13 = 69 \cdot 13 \bmod 100 = 897 = 97 \bmod 100$.

By computation we find that $13^{20} = 1 \bmod 100$. From this we have $13^{20t} = 1 \bmod 100$: $13^{20t} = (13^{20})^t = 1^t \bmod 100 = 1$. This means we are interested in the remainder mod 20 of the exponent n of 13. If $n = 20t + r$, $13^n = (13^{20})^t 13^r = 13^r \bmod 100$.

The exponent of 13 is a power of 12, 12^m, and again we see that these seem to repeat with period 4: $12, 4, 8, 16, 12, 4, \ldots$. To prove that they do indeed repeat, the previous proof for powers of 13 does not work because 1 is not in the list. So we prove by induction that the powers of 12 repeat mod 4.

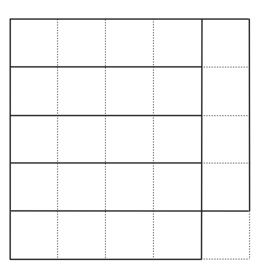
The induction statement is $12^{4t+r} = 12^r \bmod 4$. By computation it is true for $t = 1, r = 1, 2, 3, 4$. Now assume that $12^{4(t-1)+r} = 12^r \bmod 4$. Then $12^{4t+r} = 12^{4(t-1)+r+4} = 12^{4(t-1)+r}12^4 = 12^r \cdot 12^4 \bmod 4 = 12^{r-1} \cdot 12^5 = 12 \bmod 4 = 12^{r-1} \cdot 12 \bmod 4 = 12^r$.

Finally we look at $m = 11^k \bmod 4$ and we see that $11^k \bmod 4$ alternates $3, 1, 3, 1, 3, \ldots$ and is always 1 if k is even. Since k is a power of 10 it is even, so $m = 11^k = 1 \bmod 4$, $n = 12^1 = 12 \bmod 20$, and finally $13^{12} = 81 \bmod 100$, again by computation, reducing mod 100 as we go along.

4. (From the answer of Samuel Dittmer, Zionsville).

(a) We show that it is not possible to determine the numbers on an 8 by 8 chessboard from the given information, by giving two different arrangements with the same pair sums. Let the squares be colored by two colors, say red and blue, in the usual diagonal fashion. Note that any two adjacent squares, either horizontally or vertically, have one red and one blue. Place the numbers $1, 2, \ldots, 32$ in the red squares and $33, 34, \ldots, 64$ in the blue. For the second arrangement, replace each number x in a red square with $x + 32$, and each number y in a blue square with $y - 32$. Then if the sum of two adjacent squares was $x + y$, it is now $x + 32 + y - 32 = x + y$. Thus the sums for the second arrangement are the same.

(b) For a square of odd sidelength, such as 2007, it is possible to recover the values in the squares. The figure shows the case of sidelength 5. Each row has an odd number of squares, so the sum of all the entries except the last can be computed. Similarly the sum of all the entries in the last column, except for the last, can be computed. When all of these sums are added together, they give the sum of all but one entry. But the sum of all the entries in the square is the sum of the first 2007^2 integers: $(2007^2)(2007^2 + 1)/2$. By subtraction we find the entry in the bottom right corner, from which all the rest can be determined.



2007 IUPUI HIGH SCHOOL MATH CONTEST

First Prize Winner

Samuel Dittmer, 11th Grade, Zionsville Community High School. Teacher: Mrs. Jean Glore

Second Prize Winner

Ruofan Xia, 10th Grade, Carmel High School. Teacher: Mrs. Janice Mitchener

Third Prize Winner

Raphael-Joel Lim, 10th Grade, Pike High School. Teacher: Mr. John Dorsey

Fourth Place Winners

Pablo Davila, 11th Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Bosen Liu, 12th Grade, Hamilton Southeastern. Teacher: Mrs. Letitia McCallister.

Dewei Yang, 10th Grade, Carmel High School. Teacher: Mr. Matthew Wernke.

Tianyi Zhang, 10th Grade, Carmel High School. Teacher: Mrs. Kathie Freed.

Fifth Place Winners

Jonathan Bons, 11th Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Christina Cook, 11th Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Ian Goller, 10th Grade, Fishers High School. Teacher: Mrs. Kathleen Robeson.

Sarah Hill, 9th Grade, Fishers High School. Teacher: Mr. John Drozd.

Yingxue Li, 10th Grade, Carmel High School. Teacher: Mrs. Kathie Freed.

Joel Lugo, 11th Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Parth Patel, 11th Grade, Fishers High School. Teacher: Mr. John Drozd.

Winston Schwalm, 11th Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Andy Vissing, 10th Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Honorable Mention Winners

Kevin Blessinger, 10th Grade, Brownsburg High School. Teacher: Mrs. Micah Knobel.

Sean Brock, 11th Grade, Brownsburg High School. Teacher: Mrs. Micah Knobel.

Adrienne Bruce, 11th Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Andrew Campbell, 10^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Ryan Carr, 11^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Tyler Derr, 11^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Erin Diamond, 11^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

David Fifer, 9^{th} Grade, Brownsburg High School. Teacher: Mrs. Micah Knobel.

Amanda Fiorini, 11^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Callen Gustafson, 12^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Letitia McCallister.

Kathryn Hockemeyer, 10^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Elizabeth Horne, 10^{th} Grade, North Central High School. Teacher: Mrs. Sheila Varchetti.

Jacquelin Kammeyer, 11^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Britt Koehnlein, 11^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Tim McCall, 10^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Sean Morrissey, 12^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Austin Mudd, 11^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Matthew O'Bannon, 9^{th} Grade, Brownsburg High School. Teacher: Mrs Micah Knobel.

Fatemah Parvin-Nejad, 10^{th} Grade, Fishers High School, Mrs. Kathleen Robeson.

Nikhil Patel, 11^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Thomas Rothrock, 9^{th} Grade, Brownsburg High School. Teacher: Mrs. Micah Knobel.

Julia Strzeszkowski, 10^{th} Grade, Hamilton Southeastern. Teacher: Mrs. Susan Wong.

Hobey Tam, 11^{th} Grade, Fishers High School. Teacher: Mrs. Kathleen Robeson.

Emily Wang, 11^{th} Grade, Fishers High School. Teacher: Mr. John Drozd.