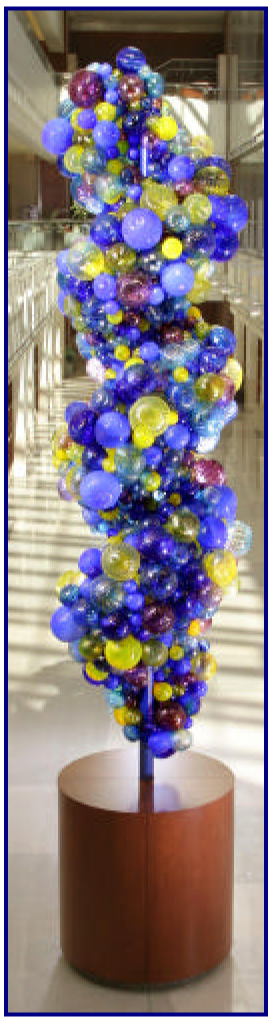
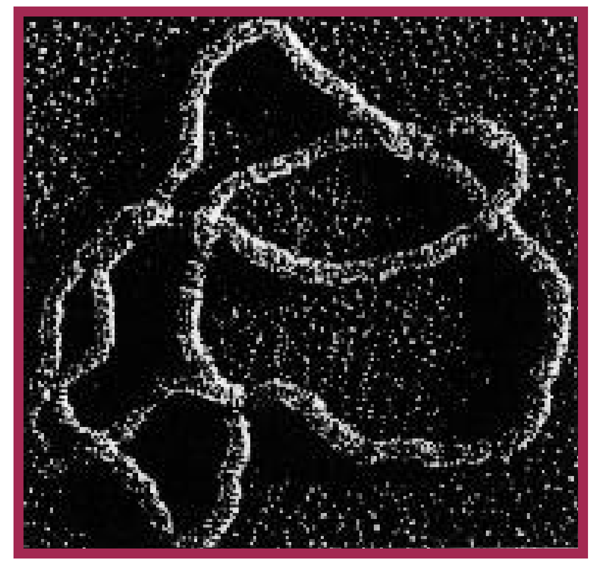
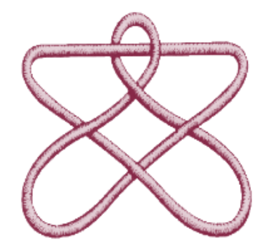


The Indiana University School of Medicine DNA Tower, a sculpture by artist Dale Chihuly, was commissioned to commemorate the centennial of the School and the 50th anniversary of the discovery of DNA by IU alumnus James D. Watson and colleague Francis Crick. More than a thousand glass balls cluster a double-helical armature of steel that supports the "twisted ladder" design. The sculpture is in the lobby of the Van Nuys Medical Science Building on the IUPUI campus. Photographs courtesy of Indiana University's Office of Visual Media.



Bacterial DNA can sometimes form in the shape of a knot.

Above: DNA electron micrograph
Below: Stevedore's Knot
Images courtesy of N. Cozzarelli



IUPUI/Roche Diagnostics
2004 High School Mathematics Contest

Presented by

The IUPUI Department of Mathematical Sciences and Roche Diagnostics

STUDENT PRIZES:

? First prize $500 ? 5 second prizes $300 each ? 10 third prizes $150 each
Honorable mentions will receive a gift and all entrants will receive certificates honoring their participation.

MATHEMATICS DEPARTMENT PRIZES:

The school whose students have the best overall performance will be awarded a cash prize of $500 and a traveling trophy. In addition, the judges will recognize two other participating schools with awards of $500 each. Schools awarded the trophy in the past:

? Hamilton Southeastern, 2003 ? Ben Davis, 2001 ? Roncalli, 1999
? Hamilton Southeastern, 2002 ? Carmel, 2000 ? Brebeuf Jesuit, 1998.

CEREMONY:

Prizewinners and honorable mentions will be invited to an awards ceremony at IUPUI on Friday, May 14, 2004 from 4:00 to 6:30 PM. Parents and teachers will also be invited. The program will feature the awards presentation, refreshments and a talk about an application of mathematics to genetics.

ELIGIBILITY:

This contest is open to high school students (grades 9-12) in 14 counties of Central Indiana: Bartholomew, Boone, Brown, Clinton, Hamilton, Hancock, Hendricks, Howard, Johnson, Madison, Marion, Morgan, Shelby and Tipton.

ENTRIES:

Submit your entry by Friday, April 16, 2004. You may obtain a copy of the questions, instructions for mailing your entry and the cover sheet from your math teacher or the contest website. Solve the questions, giving your reasoning, not just the answers. Entries will be judged by professors in the IUPUI Department of Mathematical Sciences. Judging will be based on elegance of solution as well as correctness.

QUESTIONS:

1. A sequence $a_1, a_2, a_3, \ldots$ has the property that $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$. If the sum of the first 1492 terms is 1865 and the sum of the first 1865 terms is 1492, what is the sum of the first 1492+1865 terms?

2. A palindrome number is a whole number that reads the same backwards as forwards – for example 13531. Find the set of all palindrome numbers not containing the digit 0, whose squares are also palindromes. What if the digit 0 is allowed?

3. Let $ABCD$ be a convex quadrilateral. Show that it has the property that there is a point P inside such that the areas of the 4 triangles $\Delta APB, \Delta BPC, \Delta CPD$ and ΔDPA are equal if and only if one of the diagonals of the quadrilateral bisects its area.

4. Your teacher is finding the decimal expansion of $\frac{m}{n}$, where $n \leq 100$. In the process he or she obtains the consecutive digits 167 somewhere to the right of the decimal point. Prove that he or she has made a mistake.

5. Write an essay of 500 to 700 words (complete with bibliography) on an application of mathematics to genetics.

CONTACT INFORMATION: www.math.iupui.edu/contest

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Special thanks to:
Drs. Marvin L. Bittinger,
Jeffrey X. Watt and Conrad Crown

Roche Diagnostics

Answers to Questions, 2004

1. A sequence $a_1, a_2, a_3, \ldots$ has the property that $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$. If the sum of the first 1492 terms is 1865 and the sum of the first 1865 terms is 1492, what is the sum of the first 1492+1865 terms?

Using the formula, $a_3 = a_2 - a_1$, $a_4 = -a_1$, $a_5 = -a_2$, $a_6 = -a_2 + a_1$ and $a_7 = a_1$. The sequence then repeats in periods of 6. The partial sums $s_n = a_1 + \cdots + a_n$ also repeat: $s_0 = 0$, $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = 2a_2$, $s_4 = 2a_2 - a_1$, $s_5 = a_2 - a_1$ and $s_6 = 0$. Since 1492 is equivalent to 4 mod 6 and 1865 is equivalent to 5 mod 6 we have

$$s_{1492} = s_4 = 2a_2 - a_1 = 1865$$
$$s_{1865} = s_5 = a_2 - a_1 = 1492$$

Hence $a_2 = 373$. So $s_{1492+1865} = s_3 = 2a_2 = 746$.

2. A palindrome number is a whole number that reads the same backwards as forwards – for example 13531. Find the set of palindrome numbers not containing the digit 0, whose square is also a palindrome. What if the digit 0 is allowed?

Two of the entrants, Thomas Pollum from Cathedral High School and Brian Thomas from Hamilton Southeastern High School independently discovered and proved the following Theorem. To the best knowledge of the contest committee this is a new result.

Theorem. The square of a palindrome a is again a palindrome if and only if the sum of squares of the digits of a is less than 10.

This gives the following list of solutions of palindromes without 0:

$$\{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111, 2, 22, 121, 212, 11211, 3\}$$

If 0 is allowed then any solution can be obtained from the above numbers, excluding the single digit numbers, 1, 2, 3, by insertion of zeros between the digits as long as the number remains a palindrome.

The proof of the Theorem requires two simple lemmas. For these, assume that a is a palindrome number such that a^2 is also a palindrome. Let $a = a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_0$, so that $a_n, a_{n-1}, \ldots, a_0$ are the first, second, ..., the last digits of a from the left. Since a is a palindrome number, $a_j = a_{n-j}$.

Lemma 1. The first digit of a, can be 1, 2, or 3, only.

Proof. Consider the squares, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$. If a begins and ends with 4, then the last digit of a^2 is 6. As $4 \times 10^k < a < 5 \times 10^k$ for some k, $16 \times 10^{2k} < a^2 < 25 \times 10^{2k}$ so the first digit of a^2 is 1 or 2. Hence a^2 is not a palindrome. A similar argument is valid for all digits greater than 3.

Lemma 2. When we apply the long multiplication algorithm to multiply a by a, there is no carryover in any digit.

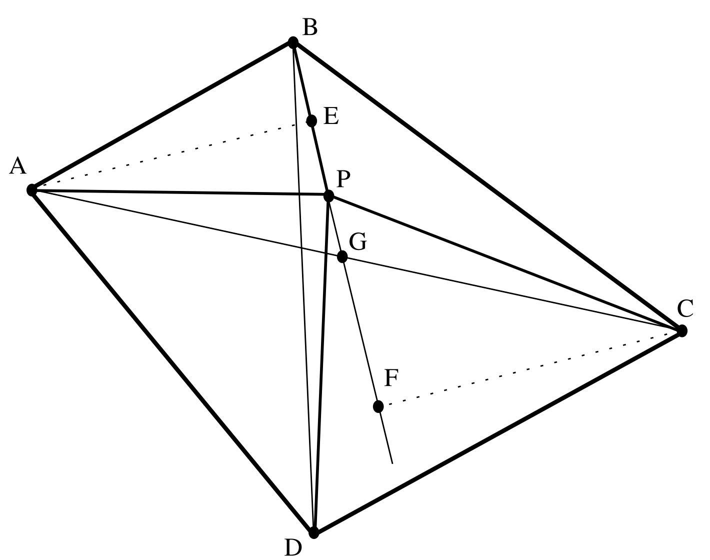
Proof. The proof is by induction on the digit number from the left. Assume that there is a carryover to the first digit in a^2. By Lemma 1 the first digit of a is 1, 2, or 3. If it is 1 and there is a carryover to the first digit in a^2, then using an inequality like that in the proof of Lemma 1, the first digit of a^2 is 2 or 3, while the last digit of a^2 is 1. This is impossible, because a^2 is a palindrome. A similar argument applies when the first digit of a is 2 or 3. Thus, there is no carryover to the first digit in a^2. In our notation, the number a consists of $n+1$ digits. Since there is no carryover to the first digit in a^2, the number a^2 consists of $2n+1$ digits.

Assume now that there is no carryover to the first k digits, where $k \leq n$. Let us show that there is no carryover to the $(k+1)$-st digit. Suppose that there is. Without the carryover, the $(k+1)$-st column sum in the long division is less or equal than 9, because there is no carryover to the left of this column by the induction assumption. The $(k+1)$-st column sum is the same as the one in the $(k+1)$-st column from the right. Therefore, any carry over to the $(k+1)$-st column would imply that a^2 is not a palindrome. This proves Lemma 2.

Proof of the Theorem. If a and a^2 are both palindromes, consider the middle column in the multiplication. Its sum is $a_n a_0 + \ldots + a_0 a_n = a_0^2 + \ldots + a_n^2$. By Lemma 2 this sum is less than 10. This proves the necessary part of the Theorem.

To prove the sufficient part, all palindromes a with $a_0^2 + \ldots + a_n^2 < 10$ are given in the list above. In each case there is no carryover in the multiplication of a by a and a^2 is also a palindrome.

3. Let $ABCD$ be a convex quadrilateral. Show that it has the property that there is a point P inside such that the areas of the 4 triangles $\Delta APB, \Delta BPC, \Delta CPD$ and ΔDPA are equal if and only if one of the diagonals of the quadrilateral bisects its area.



Assume first that diagonal $\overline{AC}$ bisects the area. Then $\triangle ABC$ and $\triangle ADC$ have the same base $\overline{AC}$, therefore congruent altitudes. If P is chosen to be the midpoint of $\overline{AC}$, the four triangles formed will have the same area.

Conversely suppose there is a point P such that the four areas are equal. If P lies on one of the diagonals, say $\overline{AC}$, then clearly $\overline{AC}$ bisects the area of $ABCD$. It remains to show that if P does not lie on diagonal $\overline{BD}$ then it *must* lie on $\overline{AC}$.

First we show that P lies on the median of $\triangle ABC$. Since triangles $\triangle APB$ and $\triangle CPB$ share the base $\overline{BP}$, they must have congruent altitudes $\overline{AE}$ and $\overline{CF}$ (see the figure). The lines $\overline{AE}$ and $\overline{CF}$ are parallel and $\overline{AC}$ is a transversal, so $\angle EAC = \angle ACF$. So the triangles $\triangle EAG$ and $\triangle FCG$ are congruent, making G the midpoint of $\overline{AC}$. So P lies on the median $\overline{BG}$.

Similarly P lies on the median of $\triangle ADC$. The only point that lies on the medians of both triangles is G, so $P = G$ lies on $\overline{AC}$.

4. Your teacher is finding the decimal expansion of $\frac{m}{n}$, where $n \leq 100$. In the process he or she obtains the consecutive digits 167 somewhere to the right of the decimal point. Prove that he or she has made a mistake.

If nonzero m and n are not both positive integers, the sequence 167 could have arisen, say, from $m = 0.167$ and $n = 1$ or from $m = 167$ and $n = -1000$.

Otherwise, consider only positive integers m and n. For some integers p and q, m/n is bounded in its decimal expansion by $p.q167 \leq m/n < p.q168$.

Let k be the number of digits of q. Then for integer $r = (10^k)(p.q)$, $r.167 \leq (10^k)m/n < r.168$. Therefore $(r.167)n \leq (10^k)m < (r.168)n$.

Subtract rn and multiply by 6 to get integer $L = 6[(10^k)m - rn]$ bounded by $(1.002)n \leq L < (1.008)n$. Integer L is at least $n + 1$. Thus $(1.008)n$ must exceed $n + 1$. Hence $n \geq 126$. This contradicts $n \leq 100$. QED.

L is a multiple of 6, so 131 is the smallest n creating sequential digits 167. And in fact, $22/131 = 0.1679389\ldots$. Decimal expansions contain many such prime-valued digit sequences.

2004 IUPUI High School Math Contest Winners

First Prize Winner

Charles Tam, Sophomore, North Central High School, Teacher: Mr. Paul Brown

Second Prize Winners

Hao Yang, Freshman, Carmel High School, Teacher: Mrs. Jennifer Holmberg
Thomas Pollom, Junior, Cathedral High School, Teacher: Mrs. Lisa Ford
Nan Lin, Sophomore, Ben Davis High School, Teacher: Mrs. Sherry Aramer
Dawen Li, Freshman, Carmel High School, Teacher: Mrs. Nancy Schulenburg
Yizheng He, Sophomore, Carmel High School, Teacher: Mrs. Nancy Schulenburg

Third Prize Winners

LiLi Xu, Senior, Carmel High School, Teacher: Mrs. Nancy Schulenburg
Brian White, Junior, Hamilton Southeastern, Teacher: Mrs. Susan Wong
Bo Wang, Junior, Hamilton Southeastern, Teacher: Mrs. Nancy Schulenburg
Brian Thomas, Sophomore, Hamilton Southeastern, Teacher: Mrs. Cindy Cooper
Vanessa Stone, Senior, Brown County High School, Teacher: Mr. Matt Noriega
Brad Rogers, Senior, Kokomo High School, Teacher: Mrs. Shellie Myers
Carlin Ma, Sophomore, Carmel High School, Teacher: Mrs. Kathy Freed
Payton Lee, Sophomore, Carmel High School, Teacher: Mrs. Nancy Schulenburg
Thomas Burke, Junior, Hamilton Southeastern, Teacher: Mrs. Susan Wong
Walter Bruen, Sophomore, Brebeur Jesuit Prep School, Teacher: Mr. Ronald Ireland

Honorable Mention Winners

Jeff Alstott, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Christopher Bennett, Sophomore, Heritage Christian, Teacher: Mrs. Grace Chitty
Stephen Cogswell, Freshman, Hamilton Southeastern, Teacher: Mrs. Louise Werner
Brittany Cordes, Senior, Hamilton Southeastern, Teacher: Mr. John Drozd
Ricardo Dariott, Junior, Hamilton Southeastern, Teacher: Mrs. Susan Wong
Maureen Early, Freshman, Brownsburg High School, Teacher: Mrs. Ada Stucky
Eric Fisher, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Aaron Goldman, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Michelle Greco, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Joshua Harper-Hartzell, Freshman, Hamilton Southeastern, Teacher: Mr. Ryan Taylor
Chad Haneline, Senior, Hamilton Southeastern, Teacher: Mrs. Susan Wong
William Baird, Senior, Warren Central High School, Teacher: Mr. Steven Landy
Jun He, Junior, Hamilton Southeastern, Teacher: Mrs. Susan Wong
Aaron Hershberger, Junior, Hamilton Southeastern, Teacher: Mrs. Susan Wong
Brittney Jennings, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Kenneth Kang, Freshman, Carmel High School, Teacher: Mrs. Mitchener
Laura Kinsley, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister

Honorable Mention Winners

Megan Knight, Junior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Kathryn Krengel, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Uday Kumar, Sophomore, Carmel High School, Teacher: Mrs. Nancy Schulenburg
Brittany Lash, Senior, Hamilton Southeastern, Teacher: Mr. John Drozd
Maren Maxis, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Alex Mills, Freshman, Hamilton Southeastern, Teacher: Mr. Ryan Taylor
Jaclyn Myers, Sophomore, Hamilton Southeastern, Teacher: Mrs. Cindy Cooper
Sonia Nevrekar, Junior, Hamilton Southeastern, Teacher: Mrs. Susan Wong
Fred Pai, Freshman, Hamilton Southeastern, Teacher: Mrs. Louise Werner
Laura Peregrim, Junior, Hamilton Southeastern, Teacher: Mrs. Susan Wong
Shenil Shah, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Zachery Smith, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Morgan Smith, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Julia Song, Freshman, Hamilton Southeastern, Teacher: Mrs. Louise Werner
Fredrick Starks, Freshman, Brownsburg High School, Teacher: Mrs. Ada Stucky
Angela Stevenson, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Samuel Tucker, Freshman, North Central High School, Teacher: Mrs. Sheila Varchetti
Danielle Veillette, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Luke Xie, Sophomore, Carmel High School, Teacher: Mrs. Nancy Schulenburg
Rachel Zeller, Senior, Hamilton Southeastern, Teacher: Mrs. Letitia McCallister
Hans Zhao, Sophomore, Carmel High School, Mrs. Nancy Schulenburg