MATHEMATICS IN ARCHITECTURE. The St. Louis Arch welcomes travelers to the gateway to the west and honors the explorers of the Northwest Territory. It was designed by the Finnish-born architect, Eero Saarinen, and was constructed from 1961-1965. This catenary arch, a 630 ft. high gracefully sweeping tapered curve of stainless steel, is the tallest man-made monument in the United States. *Poster image courtesy of Jefferson National Expansion Memorial/National Park Service*

IUPUI / Roche Diagnostics
2002 High School Mathematics Contest

Presented by
The IUPUI Department of Mathematical Sciences and Roche Diagnostics Corporation

PRIZES:

First prize — $500 15 second prizes — $200 each
In addition, one-year scholarships to IUPUI will be awarded to top performing Juniors and Seniors. Five of the $200 prizes will reward especially elegant submissions for each individual problem. Honorable mentions will receive a popular book or video on mathematics, and all entrants will receive certificates honoring their participation. The school whose students have the best overall performance will be awarded a traveling trophy. Schools awarded the trophy in the past:

Brebeuf, 1998	Carmel, 2000
Roncalli, 1999	Ben Davis, 2001

CEREMONY:

Prizewinners and honorable mentions will be invited to an awards ceremony at IUPUI on Friday, May 10, 2002, from 4:00 to 6:30 PM. Parents and teachers will also be invited. The program will feature the awards presentation, catered refreshments and a talk on applications of mathematics in architecture.

ENTRIES:

Submit your entry by Friday, April 12, 2002. You may obtain a copy of the questions, the instructions for mailing your entry and the cover sheet from your math teacher (or by contacting the IUPUI Math Department). Solve the questions, giving your reasoning, not just the answer. Entries will be judged by professors in the IUPUI Department of Mathematical Sciences. Judging will be based on elegance of solution as well as correctness.

CONTACT INFORMATION: www.math.iupui.edu/contest

IUPUI/Roche Diagnostics Mathematics Contest
Department of Mathematical Sciences
402 North Blackford Street
Indianapolis, IN 46202-3216
(317) 274-MATH contest@math.iupui.edu

QUESTIONS:

1. Evaluate the following:

$$\frac{(2\cdot 5+2)(4\cdot 7+2)(6\cdot 9+2)(8\cdot 11+2)\ldots(1998\cdot 2001+2)}{(1\cdot 4+2)(3\cdot 6+2)(5\cdot 8+2)(7\cdot 10+2)\ldots(1997\cdot 2000+2)}$$

2. Let a,b,c,d be the lengths of consecutive sides of a quadrilateral, S its area. Prove that

$$S \le \frac{(a+c)(b+d)}{4}$$

3. There are four straight lines in the plane. No two are parallel and no three meet at one point. Along each line a pedestrian walks at a constant speed. It is known that the first pedestrian meets the second, the third and the fourth ones, and the second pedestrian meets the third and fourth ones. Prove that the third pedestrian meets the fourth one.

4. There are 2002 plastic disks on a table and a large supply of extra disks. Some of the disks are colored red, some white and some blue. At each step you select from the table any two disks of different colors and exchange them with two disks of the third color from the extra supply (so that there are still 2002 disks on the table). Prove that it is possible after a finite number of steps to have all of the disks on the table of the same color. Moreover, show that the final color is independent of your sequence of steps.

5. Write an essay of 500 to 700 words (complete with bibliography) on an application of mathematics to architecture.

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The Point

Thanks to: Drs. Marvin L. Bittinger, Jyotirmoy Sarkar, Jeffrey X. Watt and Conrad Crown

Solutions
2002 IUPUI/Roche Diagnostics High School Math Contest

1. Evaluate the following:

$$\frac{(2\cdot 5+2)(4\cdot 7+2)(6\cdot 9+2)(8\cdot 11+2)\cdots(1998\cdot 2001+2)}{(1\cdot 4+2)(3\cdot 6+2)(5\cdot 8+2)(7\cdot 10+2)\cdots(1997\cdot 2000+2)}$$

Solution: The nth factor is

$$\frac{2n(2n+3)+2}{(2n-1)(2n+2)+2}=\frac{n+1}{n}$$

with $n=1\ldots 999$. So the formula can be rewritten

$$\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{1000}{999}$$

which telescopes to the value 1000.

2. Let a,b,c,d be the lengths of consecutive sides of a quadrilateral, S its area. Prove that

$$S\le\frac{(a+c)(b+d)}{4}$$

Solution: First notice that if a and b are 2 sides of a triangle (see Figure A), the area is no larger than $\frac{1}{2}ab$: Area $=\frac{1}{2}ah\le\frac{1}{2}ab$.

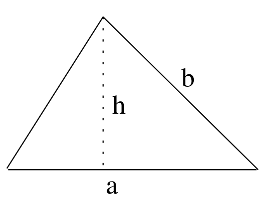

Fig. A

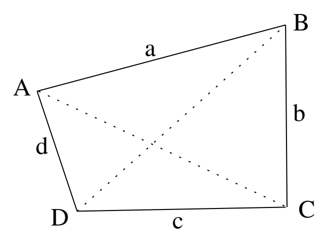

Fig. B

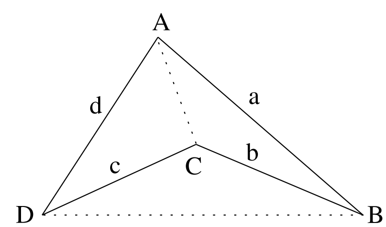

Fig. C

In the case of a convex quadrilateral, draw its 2 diagonals (see Figure B). By the above we have

$$2S=\text{Area}(\triangle ABC)+\text{Area}(\triangle BCD)+\text{Area}(\triangle CDA)+\text{Area}(\triangle DAB)\le\frac{1}{2}ab+\frac{1}{2}bc+\frac{1}{2}cd+\frac{1}{2}ad$$

This simplifies to the stated result.

If the quadrilateral is nonconvex, again draw the two diagonals (see Figure C). We have

$$\begin{aligned}S&=\text{Area}(\triangle ABC)+\text{Area}(\triangle CDA)\\&=\text{Area}(\triangle DAB)-\text{Area}(\triangle BCD)\end{aligned}$$

So

$$\begin{aligned} 2S &= \text{Area}(\triangle ABC) + \text{Area}(\triangle CDA) + \text{Area}(\triangle DAB) - \text{Area}(\triangle BCD) \\ &\le \text{Area}(\triangle ABC) + \text{Area}(\triangle CDA) + \text{Area}(\triangle DAB) + \text{Area}(\triangle BCD) \end{aligned}$$

Now finish as before.

3. There are four straight lines in the plane. No two are parallel and no three meet at one point. Along each line a pedestrian walks at a constant speed. It is known that the first pedestrian meets the second, the third and the fourth ones, and the second pedestrian meets the third and fourth ones. Prove that the third pedestrian meets the fourth one.

It is not enough just to show that the 4 paths intersect; the pedestrians on two paths meet only if they reach the intersection point at the same time.

Solution of Scott Dial, Ben Davis High School: Let L_1, L_2, L_3, L_4 denote the 4 lines and P_1, P_2, P_3, P_4 the 4 pedestrians. Extend an axis perpendicular to their plane representing an axis of time. Each pedestrian travels at a constant speed, so the graphs in 3-space of their motion are straight lines M_1, M_2, M_3, M_4. Line M_i projects to line L_i. If (x, y, t) is on M_i, it means P_i passed through (x, y) at time t. Since P_1 and P_2 met, at the time of their encounter they were located at the same planar point. Therefore lines M_1 and M_2 intersect. Since P_3 met both P_1 and P_2, M_3 intersects both M_1 and M_2. Therefore they all lie in the same plane. The same argument applies to P_4, so all 4 lines M_1, M_2, M_3, M_4 lie in the same plane. Lines M_3 and M_4 could not be parallel, since L_3 and L_4 intersect. The fact that the lines M_3 and M_4 intersect means that P_3 and P_4 were at the same planar point at some point in time, which means they have indeed met.

4. There are 2002 plastic disks on a table and a large supply of extra disks. Some of the disks are colored red, some white and some blue. At each step you select from the table any two disks of different colors and exchange them with two disks of the third color from the extra supply (so that there are still 2002 disks on the table). Prove that it is possible after a finite number of steps to have all of the disks on the table of the same color. Moreover, show that the final color is independent of your sequence of steps.

Solution: Let r, w, and b be the numbers of red, white and blue chips.

First note that exactly 2 of these are equivalent mod 3. They could not all be equivalent mod 3 or their sum, 2002, would be divisible by 3 and it's not. They could not all be different mod 3, say $3k+1, 3m, 3n-1$ or again their sum would be divisible by 3.

Let a, b, c be r, w, b in some order, so that $a \equiv b \bmod 3$ and $a \ge b$.

At each step we convert (a, b, c) to one of $(a-1, b-1, c+2)$, $(a-1, b+2, c-1)$ or $(a+2, b-1, c-1)$. In any case, the numbers in the first 2 places remain equivalent mod 3.

By performing the step $(a, b, c) \longrightarrow (a-1, b-1, c+2)$ b times, we obtain the triple $(a', 0, c')$, with $a' \equiv 0 \bmod 3$. If $a' = 0$ we are done. Otherwise convert to $(a'-1, 2, c'-1)$ and then do the first step twice more to obtain $(a'', 0, c'')$, where $a'' = a' - 3$. Repeat until $(0, 0, 2002)$ is obtained.

The final color is the one represented by c originally, the one not equivalent to a, b.

First Prize Winner

- Jonathan Steven Landy, Warren Central High School.

Second Prize Winners

Problem 1:

- Sheryl Pai, Hamilton Southeastern High School.

Problem 2:

- Nan Tian, Carmel High School.

Problem 3:

- Scott Dial, Ben Davis High School.

Problem 4:

- Dave Bauman, Roncalli High School.

Problem 5:

- Irene Sun, Ben Davis High School.

Overall:

- Aileen Chen, Carmel High School.
- Mina Dimitri, Carmel High School.
- Jonathan Eessalu, Zionsville Community High School.
- Amy Hoffman, Carmel High School.
- Imade Imasuen, Northview Middle School.
- Jixin Li, North Central High School.
- Tasha Matsumoto, Hamilton Southeastern High School.
- Patrick Mihelich, Park Tudor High School.
- Christopher Murphy, Carmel High School.
- Laurel Schrementi, Carmel High School.

Honorable Mention Winners

- Charlene Allison, Hamilton Southeastern High School.
- Janice Cai, Hamilton Southeastern High School.
- Henry Chou, Hamilton Southeastern High School.
- Melissa Dere, Carmel High School.
- Jeffery Derrenberger, Hamilton Southeastern High School.
- Thomas Herbert, Hamilton Southeastern High School.
- Brittany Lash, Hamilton Southeastern High School.
- Stephanie McGowan, Brebeuf Jesuit Preparatory School.
- Jeff Ostendorf, Brebeuf Jesuit Preparatory School.
- Christopher Reid, Hamilton Southeastern High School.

- Talwyn Scudi, Arsenal Technical High School.
- Jacob Teitgen, Hamilton Southeastern High School.
- Matthew Wiesen, Carmel High School.
- Nicole Wright, Hamilton Southeastern High School.