IUPUI / TMMI
2000 High School Mathematics Contest

Presented by
The IUPUI Department of Mathematical Sciences and Toyota Motor Manufacturing, Indiana

PRIZES:

First prize — \$500 15 second prizes — \$200 each
Honorable mentions — popular book or video on mathematics.

Five of the \$200 prizes will reward especially elegant submissions for each individual problem. All entrants will receive certificates honoring their participation. The school whose students have the best overall performance will be awarded a traveling trophy.

CEREMONY:

Prizewinners and honorable mentions will be invited to an awards ceremony at IUPUI on Friday, May 12, 2000, from 4 to 6 PM. Parents and teachers will also be invited. The program will feature the awards presentation, a talk on applications of mathematics in environmental sciences, and catered refreshments.

ENTRIES:

Submit your entry by Friday, April 14, 2000. You may obtain a copy of the questions, the instructions for mailing your entry and the cover sheet from your math teacher (or by contacting the IUPUI Math Department). Solve the questions, giving your reasoning, not just the answer. Entries will be judged by professors in the IUPUI Department of Mathematical Sciences. Judging will be based on elegance of solution as well as correctness.

CONTACT INFORMATION: www.math.iupui.edu/contest

IUPUI/TMMI Mathematics Contest
Department of Mathematical Sciences
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Indianapolis, IN 46202-3216
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QUESTIONS:

1. A rook stands on the lower left square of a chessboard. Is there a path that takes the rook through every square of the chessboard once and only once, and that ends at the upper right square?

2. Find positive integers $x_1, x_2, \ldots, x_{1999}, x_{2000}$ such that

$$1-\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\ldots+\cfrac{1}{1999+\cfrac{1}{2000}}}}}=\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{1}{x_3+\ldots+\cfrac{1}{x_{1999}+\cfrac{1}{x_{2000}}}}}}$$

3. Find all integer solutions of the equation

$$x^3 = 6y^3 + 20z^3.$$

4. Three circles with the same radius and centers O_1, O_2 and O_3 intersect at a given point A. Let A_1, A_2 and A_3 be the other intersection points. Prove that $\Delta O_1O_2O_3$ is congruent to $\Delta A_1A_2A_3$.

5. Write an essay of 400 to 600 words (complete with bibliography) on an application of mathematics to the study of the environment.

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SPECIAL THANKS TO:

Professor Marvin L. Bittinger Professor Emeritus Conrad Crown Professor Jeffrey X. Watt

Answers to the 2000 IUPUI/TMMI Mathematics Contest

1. A rook stands on the lower left square of a chessboard. Is there a path that takes the rook through every square of the chessboard once and only once, and that ends at the upper right square?

Solution. No such path exists. If the squares are alternately black and white, the beginning and ending squares are the same color. Any path can be broken up into short paths that advance only one square, which changes color each time. Since 63 moves are required, the change of color occurs an odd number of times, leaving the rook on the opposite color from which it started.

2. Find positive integers $x_1, x_2, \ldots, x_{1999}, x_{2000}$ such that

$$1 - \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \ldots + \cfrac{1}{1999 + \cfrac{1}{2000}}}}} = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \ldots + \cfrac{1}{x_{1999} + \cfrac{1}{x_{2000}}}}}}$$

Solution.

By solving special cases of the equations, one conjectures that the solution is $x_1 = 1$, $x_2 = 1$, $x_3 = 3$ and from then on, $x_n = n$. Substituting these in the right side of the equation we obtain a formula we can call F:

$$F = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \ldots + \cfrac{1}{1999 + \cfrac{1}{2000}}}}}$$

To prove the conjecture that F equals the left side L of the equation, we set t equal to the common portion:

$$t = \cfrac{1}{3 + \ldots + \cfrac{1}{1999 + \cfrac{1}{2000}}}$$

The left side L becomes

$$1 - \frac{1}{2+t} = \frac{t+1}{t+2}$$

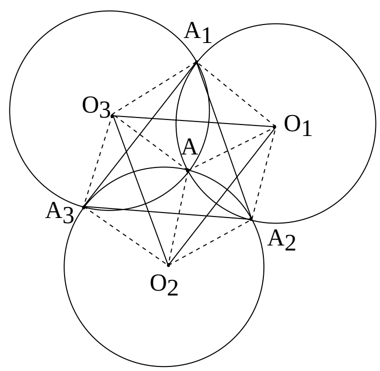
and F also reduces to the same quantity:

$$F = \frac{1}{1 + \frac{1}{1+t}} = \frac{t+1}{t+2}$$

3. Find all integer solutions of the equation $x^3 = 6y^3 + 20z^3$.

Solution. The only solution is $x = y = z = 0$. Suppose x, y, z is another solution. We can assume that x, y, z have no common factor, since if there is a common factor it can be divided out and the new triple will still be a solution. From the equation, we see that x must be an even number, say $x = 2p$. Substituting and cancelling a 2 leaves $4p^3 = 3y^3 + 10z^3$. From this we see that $3y^3$, hence y, must also be even, say $y = 2q$. Substituting and cancelling another 2 gives $2p^3 = 12q^3 + 5z^3$, so z must be even. This contradicts the assumption that x, y, z have no common factor.

4. Three circles with the same radius and centers O_1, O_2 and O_3 intersect at a given point A. Let A_1, A_2 and A_3 be the other intersection points. Prove that triangle $O_1O_2O_3$ is congruent to triangle $A_1A_2A_3$.



All of the dotted segments are radii, hence have the same length. So quadrilaterals $A_1O_1AO_3$, $A_2O_2AO_1$ and $A_3O_3AO_2$ are rhombii, hence parallelograms. Concentrating on $A_2O_2AO_1$ and $A_3O_3AO_2$, we see that segment O_1A_2 is congruent and parallel to AO_2 and hence O_3A_3. So $O_1A_2A_3O_3$ is a parallelogram and therefore side O_3O_1 is congruent to side A_3A_2. Repeating this argument in turn for the other two choices of 2 parallelograms, we find all three sides of triangle $O_1O_2O_3$ are congruent to the corresponding sides of triangle $A_1A_2A_3$. Thus the triangles are congruent.

2000 IUPUI/TMMI High School Mathematics Contest Winners

First Prize Winner

Jonathan Steven Landy, Warren Central High School, Teacher: Mrs. Gaerte

Second Prize Winners

Jack Province, North Central High School, Teacher: Mrs. Renee South
Amy Hoffman, Carmel High School, Teacher: Mrs. Jan Mitchener
Steven Linville, Franklin Community High School, Teacher: Mrs. Cheryl Flater
Brigid Marie Slinger, Brebeuf Jesuit Preparatory School, Teacher: Ms. Laycock
Sandy Ottensmann, Brebeuf Jesuit Preparatory School, Teacher: Ms. Joan Rocap
John Dionisios Aliprantis, Brebeuf Jesuit Preparatory School, Teacher: Ms. Joan Rocap
William H. Bruns, Carmel High School, Teacher: Mrs. Jan Mitchener
Aileen Chen, Carmel High School, Teacher: Jan Mitchner
Matthew Adam Fischer, Brebeuf Jesuit Preparatory School, Teacher: Ms. Laycock
Megan Konstant, Roncalli High School, Teacher: Mrs. Ramey
Anand Kulanthaivel, Brebeuf Jesuit Preparatory School, Teacher: Ms. Joan Rocap
Todd McCready, Lawrence Central High School, Teacher: Mrs. Dowden and Meinen
Christopher W. Murphy, Carmel High School, Teacher: Mrs. Jan Mitchener
Joseph Harley Teal, Brown County High School, Teacher: Mr. Dave Langell
Jeremy Daniel Tryba, Carmel High School, Teacher: Mrs. Jan Mitchener

Honorable Mention Winners

Jean Bao, Carmel High School, Teacher: Mrs. Jan Mitchener
David T. Osburn, Roncalli High School, Teacher: Mrs. Ramey
Megan Pfarr, Roncalli High School, Teacher: Mrs. Ramey
Melissa Phillips, Cardinal Ritter High School, Teacher: Mrs. Werner
Kathryn Ann Tolle, Roncalli High School, Teacher: Sr. Anne Frederick
Matt Willsey, Roncalli High School, Teacher: Sr. Anne Frederick