Math 171, Professor Ramras

Linear Algebra Practice Problems

(1) Consider the following system of linear equations in the variables x, y, and z, in which the constants a and b are real numbers.

$$\begin{aligned} 2x - 3y + 2z &= a \\ -by + 3z &= 3 \\ x - y - z &= a + b. \end{aligned}$$

For what values of a and b will the system have infinitely many solutions? A unique solution? No solutions? **Make sure to answer each part of the question.**

(2) a) Find all solutions to the linear system with the following augmented coefficient matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 3 & 1 & 2 \\ 0 & 2 & 2 & -8 \end{array}\right)$$

b) List one numerical solution to the above system of equations, and check that your solution satisfies the system.

(3) In each case, find the reduced row echelon form of the given system of equations, and describe the solution set in parametric form.

a) $$\begin{array}{rrrl} x_1 & +x_2 & +2x_3 & = 1 \\ 3x_1 & +3x_2 & +3x_3 & = 9 \\ 4x_1 & +4x_2 & +5x_3 & = 10 \end{array}$$

b) $$\begin{array}{rrrrl} 3x_1 & +2x_2 & -x_3 & +x_4 & = 0 \\ 2x_1 & & +x_3 & -2x_4 & = 7 \\ x_1 & +x_2 & -4x_3 & & = 1 \\ 4x_1 & +x_2 & 4x_3 & -x_4 & = 6 \end{array}$$

c) $$\begin{array}{rcrcrcrl} x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & = 1 \\ 3x_1 & + & 3x_2 & + & 6x_3 & + & 12x_4 & = 6 \\ 2x_1 & + & 2x_2 & + & 4x_3 & + & 7x_4 & = 3 \end{array}$$

(4) For each of the following systems of linear equations, determine if there are no solutions, a unique solution, or infinitely many solutions. If there are infinitely many solutions, find the parametric form for the solution set.

a) $$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -4 \\ 4 & 6 & 2 & -4 \end{array}\right)$$
b) $$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 4 & 4 & 2 \end{array}\right)$$

(5) In each case, describe all solutions to the linear system $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$.

a) $A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 1 & -2 & 1 & -2 \end{pmatrix}$ and $\vec{\mathbf{b}}= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

b) A is the same matrix as in part a), $\vec{\mathbf{b}}=\vec{\mathbf{0}}$.

(6) Find the parametric form for solutions to the linear system $A\,\vec{\mathbf{x}}=\vec{\mathbf{b}}$, where

$$A = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 2 & 6 & 1 & 0 \end{pmatrix} \text{ and } \vec{\mathbf{b}}= \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Does this linear system have one solution, infinitely many solutions, or no solutions?

(7) a) Describe all solutions to the linear system $A\,\vec{\mathbf{x}}= \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ in parametric form, where A is the following matrix: $A = \begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -3 & 0 & -4 \\ 0 & -1 & 3 & 1 & 2 \end{pmatrix}$

b) Let A be the matrix in part a). Find two solutions $\vec{x}_1$ and $\vec{x}_2$ to the *homogeneous* system $A\,\vec{\mathbf{x}}=\vec{0}$, so that $\vec{x}_1$ and $\vec{x}_2$ are *not* scalar multiples of one another (each of your solutions should be a vector with numerical entries).

(8) a) Consider the linear system whose augmented matrix is given by

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -2 & 3 & b \\ 2 & 0 & b & 0 \end{array}\right],$$

where b is a real number. For what numbers b will the system have a unique solution?

b) What can you say about the number of solutions to the system for other values of b?

(9) For each of the following augmented matrices, state whether the matrix is in Echelon Form, Reduced Echelon Form, or neither.

a) $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$ b) $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$ c) $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$

d) $\left(\begin{array}{ccccc|c} 1 & 7 & 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$ e) $\left(\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \end{array}\right)$ f) $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right)$

g) $\left(\begin{array}{cccc|c} 0 & 1 & -4 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right)$ h) $\left(\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right)$

(10) Find the reduced row echelon forms of the following matrices:

a) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 2 & 2 & 4 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 0 \\ -1 & -4 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 1 \\ -2 & -5 & -3 \\ 2 & 3 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 5 \end{bmatrix}$

(11) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, let $B = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$, and let $\vec{\mathbf{x}} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$.

a) Compute AB and BA.

b) Compute $A\,\vec{\mathbf{x}}$ and $B\,\vec{\mathbf{x}}$.

c) Compute $AB\,\vec{\mathbf{x}}$ in two ways: first, multiplying vector $B\,\vec{\mathbf{x}}$ (which you calculated in part b)), by A; then multiply $\vec{\mathbf{x}}$ by AB (which you calculated in part a)). Make sure you get the same answer both ways! Note: when multiplying a vector by a matrix, the matrix goes on the *left*.

d) Compute $BA\,\vec{\mathbf{x}}$ in two ways, like in part c): first, multiplying vector $A\,\vec{\mathbf{x}}$ (which you calculated in part b)), by B; then multiply $\vec{\mathbf{x}}$ by BA (which you calculated in part a)). Make sure you get the same answer both ways!

(12) For which number b does the matrix $\begin{bmatrix} 8 & b \\ 15 & 4 \end{bmatrix}$ have inverse $\begin{bmatrix} 8 & -2b \\ -30 & 16 \end{bmatrix}$?

(13) Find the inverse of the matrix $A = \begin{bmatrix} 3 & -2 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$, and use this information to solve the linear systems $A\vec{x} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ and $A\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$.

(14) The matrix $A = \begin{bmatrix} 3 & -2 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ has inverse $A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}$. Use this information to solve the linear system $A\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$. Check that your solution does in fact satisfy $A\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

(15) The matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ has inverse $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$.

Use this information to solve the linear system $A\vec{x} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$.

(16) Find the inverse of each matrix below.

a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

(17) Calculate the determinants of the following matrices, and determine whether or not each matrix is invertible.

a) $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 4 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 2 & 4 \\ 3 & 3 & 7 \\ 5 & 2 & 4 \end{bmatrix}$.

e) $\begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & -3 \\ 2 & 0 & 2 \end{bmatrix}$ e) $\begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 4 & 3 & 6 \\ 5 & 5 & 5 & 2 \\ 5 & 5 & 3 & 7 \end{bmatrix}$

(18) For each vector $\vec{\mathbf{x}}$ below, determine whether or not it is in the image of the matrix $A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 8 & 11 \\ -2 & -4 & -8 \end{bmatrix}$.

a) $\vec{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ b) $\vec{\mathbf{x}}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ c) $\vec{\mathbf{x}}_3 = \begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix}$.

(19) Find the eigenvalues of the following matrices. For each eigenvalue, describe the eigenvectors with that eigenvalue.

a) $\begin{bmatrix} 5 & -2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -4 & 3 \end{bmatrix}$.

(20) Describe the eigenvectors of matrix $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$ corresponding to the eigenvalue 2.

(21) a) Is the vector $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ an eigenvector of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$? If not, explain why not; if so, determine the corresponding eigenvalue.

b) Is the vector $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ an eigenvector of the matrix $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 2 & -1 \\ 4 & 2 & 1 \end{bmatrix}$? If not, explain why not; if so, determine the corresponding eigenvalue.

(22) Compute characteristic polynomial, the eigenvalues, and the eigenvectors of the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

(23) a) Is the vector $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ an eigenvector of the matrix $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 2 & -1 \\ 4 & 2 & 1 \end{bmatrix}$? If not, explain why not; if so, determine the corresponding eigenvalue.

b) Is $\lambda = 2$ an eigenvalue of of the matrix $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$? If not, explain why not; if so, find a corresponding eigenvector.

(24) Find all solutions to the equation $A\vec{x} = 6\vec{x}$, where $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 0 \\ 1 & 1 & 4 \end{bmatrix}$.

(25) Find all solutions to the equation $A\vec{x} = 2\vec{x}$, where $A = \begin{bmatrix} 3 & -4 & -2 \\ 1 & 5 & 5 \\ 2 & -2 & 4 \end{bmatrix}$.

(26) Find all solutions to the equation $A\vec{x} = 4\vec{x}$, where $A = \begin{bmatrix} 4 & 3 & 0 \\ 2 & 1 & 6 \\ 2 & -3 & 2 \end{bmatrix}$.

(27) Find all solutions to the equation $A\vec{x} = 3\vec{x}$, where $A = \begin{bmatrix} 4 & 4 & 3 \\ 2 & 11 & 6 \\ 3 & 12 & 12 \end{bmatrix}$.