An Extension of Brolin’s Theorem & Relevant Tools

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**Theorem (Brolin, 1965)**

If \( f(z) = z^\alpha + \ldots \) is a polynomial of degree \( \alpha \geq 2 \), then there is an exceptional set \( \mathcal{E} \) with \( \#\mathcal{E} \leq 1 \) such that if \( a \in \mathbb{C} \setminus \mathcal{E} \), then

\[
\frac{1}{\alpha^n} \sum_{f^n(z) = a} \delta_z \rightarrow \mu \text{ as } n \rightarrow \infty,
\]

where \( \mu \) is harmonic measure on the filled Julia set of \( f \).

- The limit is independent of \( a \).
- \( \mathcal{E} = \emptyset \) or, if \( f \) is affinely conjugate to \( z \mapsto z^\alpha \), \( \mathcal{E} = \{0\} \).
- This result is specific for polynomials in \( \mathbb{C} \).

**Q.** Can Brolin’s Theorem extend to other types of maps or spaces?

Yes, with additional assumptions, to:

- rational maps in \( \mathbb{P}_\mathbb{C}^1 \) by Lyubich & Freire-Lopez-Mañé [1983]
- holomorphic maps in \( \mathbb{P}_\mathbb{C}^2 \) by Favre-Jonsson [2001]
Extending Brolin’s Theorem

**Theorem (Brolin, 1965)**

If \( f(z) = z^\alpha + \ldots \) is a polynomial of degree \( \alpha \geq 2 \), then there is an exceptional set \( E \) with \( \#E \leq 1 \) such that if \( a \in \mathbb{C} \setminus E \), then

\[
\frac{1}{\alpha^n} \sum_{f^n(z) = a} \delta_z \to \mu \text{ as } n \to \infty,
\]

where \( \mu \) is harmonic measure on the filled Julia set of \( f \).

**Theorem (Favre-Jonsson, 2001)**

Let \( f = [P : Q : R] : \mathbb{P}^2_\mathbb{C} \to \mathbb{P}^2_\mathbb{C} \), where \( P, Q, R \) are homogeneous polynomials of degree \( \alpha \geq 2 \) and let \( E \) be a special set. If \( S \) is a positive closed \((1, 1)\) current on \( \mathbb{P}^2 \) with mass 1 that behaves nicely on \( E \), then

\[
\frac{1}{\alpha^n} f^n* S \to T \text{ as } n \to \infty,
\]

where \( T \) is the Green current of \( f \).
Overview

1. What is a current?

   *Focus on positive closed $(1, 1)$-currents on $\mathbb{P}^2_C$.*

2. Precise statement of extension of Brolin’s Theorem to $\mathbb{P}^2_C$

   *Focus on Theorem A of “Brolin’s Theorem for Curves in Two Complex Dimensions" by Favre-Jonsson from 2001.*

3. Some Ingredients in the proof

   *Including Hartog’s Lemma.*
What is a $p$-current?

Let $M$ be a smooth ($\mathbb{R}$) manifold of dimension $m$.

Let $D^p(M)$ be the space of smooth $p$-forms with compact support on $M$.

**Definition**

A $p$-current on $M$ is a (continuous) linear functional:

$$S : D^p(M) \to \mathbb{R}.$$  

Note: The action of $S$ on $\nu \in D^p(M)$ is often denoted $\langle S, \nu \rangle$.

Let $D'_p(M)$ be the space of $p$-currents on $M$.  

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Extension of Brolin’s Theorem
Example 1. $p$-dimensional submanifolds

Let $M$ be a smooth manifold of dimension $m$.

Let $Z \subset M$ be a closed oriented submanifold of dim $p$ and class $C^1$.

Geometrically, a $p$-current can represent integration over $Z$.

The current of integration over $Z$, $[Z]$, is a $p$-current defined by:

$$\langle [Z], u \rangle = \int_Z u, \text{ for } u \in D^p(M).$$

A $p$-current $S \in D'_p(M)$ can be expressed as a $(m - p)$-form:

$$S = \sum_{|I|=m-p} S_I dx^I,$$

where $I = (i_1, \ldots, i_{m-p})$, $dx^I = dx_{i_1} \wedge \ldots \wedge dx_{i_{m-p}}$, and $i_1 < \ldots < i_{m-p}$. 

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Extension of Brolin’s Theorem
What is a $p$-current?

**Example 2. $(m - p)$-form**

A form $\alpha \in D^{m-p}(M)$ with coefficients in $L^1_{loc}$ defines a $p$-current:

$$\langle \alpha, \phi \rangle := \int_M \alpha \wedge \phi \text{ for any } \phi \in D^p(M)$$

since $\alpha \wedge \phi \in D^m(M)$ is a volume form.

Consequently, a $p$-current $S$ acts on $p$-forms and can act as an $(m - p)$-form.

We say that $S$ has dimension $p$ and degree $m - p$. 
Extending from $\mathbb{R}$ to $\mathbb{C}$

Each complex variable, $z_j$, has 2 corresponding real variables and so we have 2 corresponding differentials. In particular, $dz_j$ and $d\bar{z}_j$.

Note that $dz_j$ is a $(1, 0)$-form and $d\bar{z}_j$ is a $(0, 1)$-form.

More generally, $\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} dz_I \wedge d\bar{z}_J$ is a $(p, q)$-form and we say that $\alpha \in D^{p, q}$.

Notation: $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi} (\bar{\partial} - \partial)$, where

$$\partial \alpha = \sum_{k, |I|=p, |J|=q} \frac{\partial \alpha_{IJ}}{dz_k} dz_k \wedge dz_I \wedge d\bar{z}_j$$

$$\bar{\partial} \alpha = \sum_{k, |I|=p, |J|=q} \frac{\partial \alpha_{IJ}}{d\bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_j.$$

It follows that $dd^c = \frac{i}{\pi} \partial \bar{\partial}$. 
What is a $(1, 1)$-current?

For simplicity, we now focus on \( M = \mathbb{P}^2_\mathbb{C} \).

Let \( z_1 \) and \( z_2 \) be local coordinates on \( \mathbb{P}^2_\mathbb{C} \).

Let \( D^{1,1}(\mathbb{P}^2_\mathbb{C}) \) be the space of smooth compactly supported $(1, 1)$-forms. Any \( \nu \in D^{1,1}(\mathbb{P}^2_\mathbb{C}) \) can be expressed as:

\[
\nu = \sum_{1 \leq j, k \leq 2} a_{jk} \, dz_j \wedge d\bar{z}_k,
\]

Definition (For \( \mathbb{P}^2_\mathbb{C} \))

A $(1, 1)$-current \( S \) is a linear functional on \( D^{1,1}(\mathbb{P}^2_\mathbb{C}) \) and can be represented as a $(1, 1)$-form with distributional coefficients.
Closed positive $(1, 1)$-currents and why they are special.

**Definition**

Let $S$ be a $(1, 1)$-current and express it as $S = i \sum S_{jk} dz_j \wedge d\bar{z}_k$. $S$ is **positive** if the distribution $\sum S_{jk} \zeta_j \bar{\zeta}_k \geq 0$ for all $\zeta \in \mathbb{C}^2$.

**Definition**

A $(1, 1)$-current $S$ is **closed** if $dS = 0$ (Recall $dS = (\partial + \overline{\partial})S$).

Why are closed positive $(1, 1)$-currents special?

**Proposition (A.4.1, Sibony – some of the proposition)**

1. *Every positive $(1, 1)$-current is representable by integration.* *(The distributional coefficients are measurable)*

2. *If $S$ is a closed positive $(1, 1)$-current, then $\forall z_0 \in M$, $\exists$ an open neighborhood $U \subset M$ of $z_0$ and a plurisubharmonic function $u$ on $U$ such that $S = dd^c u$ in $U$. (Note: $u$ is called a potential of $S$ and $dd^c = \frac{i}{\pi} \partial \overline{\partial}$)*
Let $S$ be a positive closed $(1, 1)$-current on $\mathbb{P}^2_\mathbb{C}$ and $\omega$ the standard Kahler form on $\mathbb{P}^2_\mathbb{C}$ corresponding to the Fubini-Study metric.

**Definition**

*S has unit mass if* $1 = \|S\| = \int_{\mathbb{P}^2_\mathbb{C}} S \wedge \omega$.

Let $f : \mathbb{P}^2_\mathbb{C} \to \mathbb{P}^2_\mathbb{C}$ be holomorphism of algebraic degree $\alpha \geq 2$.

$\Rightarrow f = [P : Q : R], \ P, Q, R$ homogenous degree $\alpha$ polynomials.

We are now prepared to revisit FJ’s extension of Brolin’s Theorem using more precise language.
Theorem (Favre-Jonsson, 2001)

Let \( f : \mathbb{P}^2 \to \mathbb{P}^2 \) be holomorphism of algebraic degree \( \alpha \geq 2 \).

Then \( \exists \) a set \( \mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \), where:

\( \mathcal{E}_1 \) is a totally invariant, algebraic set consisting of \( \leq 3 \) \( \mathbb{C} \)-lines &
\( \mathcal{E}_2 \) is a totally invariant (i.e., \( f^{-1}(\mathcal{E}_2) = \mathcal{E}_2 \)), finite set,

and \( \mathcal{E} \) has the following property:

If \( S \) is a positive closed \((1, 1)\) current on \( \mathbb{P}^2 \) of mass 1 such that

1. \( S \) does not change any irreducible component of \( \mathcal{E}_1 \);
2. \( S \) has a bounded local potential at each point of \( \mathcal{E}_2 \);

then we have the convergence

\[ \frac{1}{\alpha^n} f^n \ast S \to T \text{ as } n \to \infty, \]

where \( T \) is the Green current of \( f \).
Part 1 of Proof of FJ Theorem

Let $\omega$ be the Fubini-Study Kahler form on $\mathbb{P}^2_C$.

$f^*\omega$ and $\alpha \omega$ are cohomologous positive closed $(1, 1)$ currents, so there is a continuous function $u$ such that:

$$f^*\omega = \alpha \omega + dd^c u.$$

Then:

$$f^{2*}\omega = \alpha(f^*\omega) + dd^c(f^*u)$$

$$= \alpha^2 \omega + dd^c (\alpha u + u \circ f).$$

Consequently,

$$f^{n*}\omega = \alpha^n \omega + dd^c(\alpha^{n-1} u + \alpha^{n-2} u \circ f + \ldots + u \circ f^{n-1})$$

and

$$\frac{1}{\alpha^n} f^{n*}\omega = \omega + dd^c \sum_{j=1}^{n-1} \alpha^{-j} u \circ f^{j-1} \to \omega + dd^c G := T \text{ as } n \to \infty.$$
On the previous slide, we had:

\[
\frac{1}{\alpha^n} f^n \omega \to T \text{ as } n \to \infty,
\]

where \( \omega \) was the Kahler-Study form.

When can we replace \( \omega \) with a current and have the same limit?

In particular, we consider positive closed \((1, 1)\) currents of mass 1.

In their proof, FJ use that such a current may affect the size of forward iterates of a ball in \( \mathbb{P}^2 \) to determine sufficient conditions on a current to attain the above limit.
Suppose that $S$ is a positive closed $(1, 1)$-current for which limit $\star$ fails. $S$ can be written as:

$$S = \omega + dd^c u,$$

where $u \leq 0$ is the sum of a psh function and a smooth function.

Then, $\forall n \geq 0$,

$$\alpha^{-n} f^n* S = \alpha^{-n} f^n* \omega + \alpha^{-n} dd^c (u \circ f^n).$$

By assumption, $\alpha^{-n} f^n* S \not\rightarrow T$ and we know that $\alpha^{-n} f^n* \omega \rightarrow T$.

So $\alpha^{-n} dd^c (u \circ f^n) \not\rightarrow 0$. Equivalently, $\alpha^{-n} u \circ f^n \not\rightarrow 0$ in $L^1_{loc}$ since:

$$\int_{\mathbb{P}^2} \alpha^{-n} dd^c (u \circ f^n) \wedge \phi = \int_{\mathbb{P}^2} (\alpha^{-n} u \circ f^n) \wedge dd^c \phi.$$
Recall: We want to determine for which $S$, $\nu_n := \alpha^{-n} u \circ f^n \not\to 0$. 

$\{\nu_n\}$ is a sequence of subharmonic functions bounded above by 0.

**Hartog’s Lemma (In Dynamics of Rational Maps on $\mathbb{P}^k$ by Sibony)**

Let $\{v_j\}$ be a sequence of subharmonic functions on a domain $\Omega$. Suppose $\{v_j\}$ is bounded above on every compact subset $K$ of $\Omega$.

If $v_j \not\to -\infty$ on $K$, then there is a subsequence $\{v_{j_k}\}$ converging on $L^1_{loc}$ to a subharmonic function $v$. In addition,

$$\limsup_{j \to \infty} \sup_K v_j \leq \sup_K v, \text{ for all compact } K.$$ 

If $\nu_n \not\to -\infty$ on a ball $B \subset \mathbb{P}^2_{\mathbb{C}}$, then there is a subsequence $\{\nu_{n_j}\}$ that converges to subharmonic $\nu < c$, for constant $c < 0$. Then:

$$B \subset \{\nu_{n_j} = \alpha^{-n_j} u \circ f^{n_j} < c\} \Rightarrow f^{n_j}(B) \subset \{u < \alpha^{n_j} c < 0\}.$$
The rest of [FJ] is spent showing that if $S$ satisfies the properties relating to $\mathcal{E}$ from the theorem, then we cannot have:

$$f^{nj}(B) \subset \{ u < \alpha^{nj} \ c < 0 \}.$$

That is done in [FJ] by estimating the volume of $f^{nj}(B)$ from below (using dynamics) and the volume of $\{ u < c\alpha^{nj} \}$ from above (using pluripotential theory).
Brolin’s Theorem, originally for monic polynomials in $\mathbb{C}$, has extensions to rational functions in $\mathbb{P}^1_\mathbb{C}$ and to holomorphic functions in $\mathbb{P}^2_\mathbb{C}$. We focused on the latter extension.