Real Rational Surface Automorphisms

김경희(Kyounghee Kim), Florida State University

March/17-18/2022

Rational Surfaces

A rational surface is a surface birationally equivalent to a projective plane $\mathbf{P}^2$.

If X is a rational surface, then there is a sequence of blowups of a point :

$$\pi : X = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \quad \cdots \quad \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbf{P}^2$$

where $\pi_i : X_i \to X_{i-1}$ is a blowup of a point $p_i \in X_{i-1}$.

Theorem (Cantat)

Let X be a compact complex surface. If there is an automorphism f on X with $h_{top}(f) > 0$ then X must be birationally equivalent to one of the followings

- *a complex torus* $\mathbb{C}/\Lambda$
- *a K3 surface*
- *the projective plane* $\mathbf{P}^2(\mathbb{C})$

Theorem (Yomdin, Gromov)

If X is a compact complex surface and f is holomorphic, then

$$h_{top}(f) = \log \rho(f^* | H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C}))$$

where ρ is the spectral radius.

If f is an automorphism on X,

$$h_{top}(f) = \log \lambda(f)$$

where $\lambda(f) = \rho(f^*|H^{1,1}(X,\mathbb{C}) \to H^{1,1}(X,\mathbb{C}))$ is the *dynamical degree* of f.

If $f \in Aut(\mathbf{P}^2(\mathbb{C}))$, then f is linear.

$$\lambda(f) = 1 \quad \Rightarrow \quad h_{top}(f) = 0$$

If f is an automorphism on a rational surface, then we can have

$$\lambda(f) > 1$$

Theorem (Nagata)

If $F : X \to X$ is an automorphism on a rational surface X with $\lambda(f) > 1$, then there is a birational map $f : \mathbf{P}^2(\mathbb{C}) \dashrightarrow \mathbf{P}^2(\mathbb{C})$ such that $f = \pi \circ F \circ \pi^{-1}$

$$\begin{array}{ccc} X & \xrightarrow{\;F\;} & X \\ \downarrow \pi & & \downarrow \pi \\ \mathbf{P}^2(\mathbb{C}) & \overset{f}{\dashrightarrow} & \mathbf{P}^2(\mathbb{C}) \end{array}$$

Theorem (Diller-Favre,Blanc-Cantat)

F is an automorphism on a rational surface if and only if $\lambda(F)$ is a Salem number.

Dynamical degree is a birational invariant. [Diller-Favre]
Find birational maps whose dynamical degrees are Salem number.

Birational Maps on $\mathbf{P}^2$

f is a birational map on $\mathbf{P}^2$:

- $f = [f_1 : f_2 : f_3]$ where $f_i's$ are homogeneous polynomials of the same degree.
- There is a rational inverse.
- There is no common divisor of f_1, f_2, f_3.
- There are points of indeterminacy :

$$\mathcal{I}(f) = \cap_i \{f_i = 0\}$$

- There are exceptional curves which map to points.

$$\mathcal{E}(f) = \{Det(Df) = 0\}$$

Theorem (Nœther Decomposition)

If $f : \mathbf{P}^2(\mathbb{C}) \dashrightarrow \mathbf{P}^2(\mathbb{C})$ is a birational map, then f can be written as a composition of the Cremona Involution J and automorphisms on $\mathbf{P}^2(\mathbb{C})$.

$$f = L_0 \circ J \circ L_1 \circ \cdots \circ L_{k-1} \circ J \circ L_k, \quad L_j \in Aut(\mathbf{P}^2(\mathbb{C}))$$

The Cremona Involution

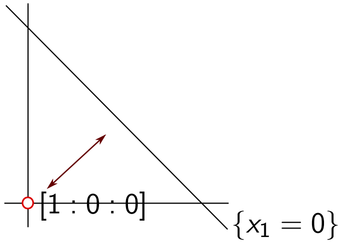
$$J : \mathbf{P}^2(\mathbb{C}) \dashrightarrow \mathbf{P}^2(\mathbb{C})$$

$$J : [x_1 : x_2 : x_3] \mapsto [\frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}] = [x_2x_3 : x_1x_3 : x_1x_2]$$

J is not defined at three points $[1:0:0], [0:1:0], [0:0:1]$

The Cremona Involution

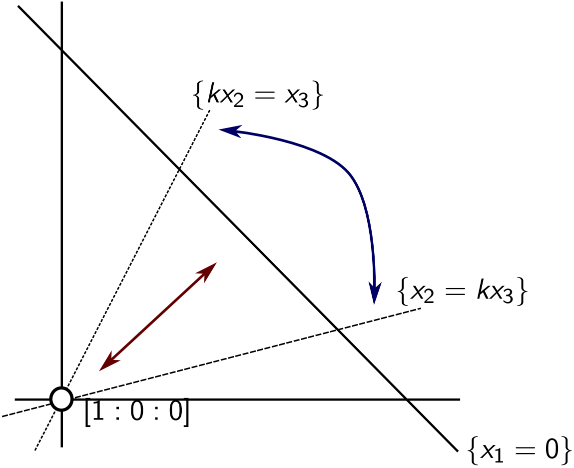
$$J : [x_1 : x_2 : x_3] \mapsto [x_2 x_3 : x_1 x_3 : x_1 x_2]$$



- the Indeterminacy locus for $J : \mathcal{I}(J) = \{e_1, e_2, e_3\}$
- the Exceptional locus for $J : \mathcal{E}(J) = \cup_i \{x_i = 0\}$

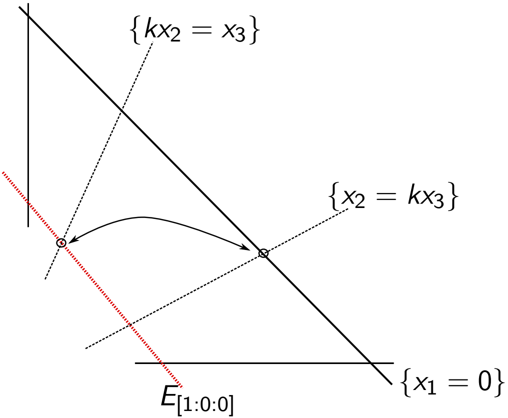
The Cremona Involution

$$J : [x_1 : x_2 : x_3] \mapsto [x_2 x_3 : x_1 x_3 : x_1 x_2]$$



The Cremona Involution

$$J : [x_1 : x_2 : x_3] \mapsto [x_2x_3 : x_1x_3 : x_1x_2] \text{ on the Blowup of } [1:0:0]$$



The Cremona involution J lifts to an automorphism on a rational surface $X = B\ell_P \mathbf{P}^2(\mathbb{C})$ where
$P = \{e_1 = [1:0:0], e_2 = [0:1:0], e_3 = [0:0:1]\}$

If $f = L \circ J$ with $L \in Aut(\mathbf{P}^2(\mathbb{C}))$,

- Each line $\{x_i = 0\}$ maps to a point $p_i = Le_i$
- Each point e_i blows up to a line $L\{x_i = 0\}$.
- If there are three positive integers and a permutation $\sigma \in S_3$ such that

$$(\mathrm{C}_\star) \quad \begin{cases} f^{n_i - 1}(p_i) = e_{\sigma(i)}, \text{and} \\ \dim f^j(p_i) = 0 \text{ for all } 0 \leq j \leq n_i - 1 \end{cases}$$

  then f lifts to an automorphism an a rational surface.

$$f = L \circ J, L = (a_{i,j}) \in GL(3, \mathbb{C}), \quad J\{x_i = 0\} = e_i$$

- f has three exceptional lines

$$p_i = f\{x_i = 0\} = \text{i-th column of } L$$

$$p_1 = e_2 \in \mathcal{I}(f) \;\Leftrightarrow\; a_{11} = a_{31} = 0$$

- The second iteration $f(p_1) = L[a_{21}a_{31} : a_{11}a_{31} : a_{11}a_{21}]$ is given by degree 3 polynomials in a_{ij}

$$f(p_1) = e_2 \;\Leftrightarrow\; \begin{cases} a_{11}a_{21}a_{31} + a_{12}a_{11}a_{31} + a_{13}a_{11}a_{21} = 0 \\ a_{31}a_{21}a_{31} + a_{32}a_{11}a_{31} + a_{33}a_{11}a_{21} = 0 \end{cases}$$

- The third iteration is given by degree 7 polynomials.
- and so on..
- To have positive entropy, we need $n_1 + n_2 + n_3 \geq 10$

- The condition $(C_\star)$ is given by 6 algebraic equations in 6 variables with HUGE degrees and finitely many non-equations.

$$(\mathrm{C}_\star) \quad \begin{cases} f^{n_i-1}(p_i) = e_{\sigma(i)}, \text{and} \\ \dim f^j(p_i) = 0 \text{ for all } 0 \leq j \leq n_i - 1 \end{cases}$$

- Are there non-empty solution sets for all possible n_1, n_2, n_3 and $\sigma \in S_3$?

- When do we have solution sets of dimension ≥ 1?
  e.g. If $n_1 = n_2 = 1, n_3 = 7$ with a cyclic permutation, the solution set has dimension 1.

Orbit data of $f = L \circ J$

Suppose there are $n_1, n_2, n_3 \in \mathbb{N}$ and a permutation $\sigma \in S_3$ such that

- $f^{n_i}\{x_i = 0\} = e_{\sigma(i)}$
- Dim $(f^j\{x_i = 0\}) = 0$ for all $1 \le j \le n_i$, $i = 1, 2, 3$

We call these numerical information **Orbit data** for f.

Theorem (Nagata)

Suppose $F : X \to X$ is an automorphism with positive entropy on a rational surface X.
Then there is a natural identification between the induced action $F^* : H^{1,1}(X) \to H^{1,1}(X)$ and an element of a Coxeter group generated by certain reflections.

The orbit data describe the induced action of, its lift, i.e. the corresponding element in a Coxeter group.

Rational surface automorphisms with positive entropy: Construction

To have positive entropy, we need $n_1 + n_2 + n_3 \geq 10$.

Using invariant curves

- McMullen : Existence of maps with $n_1 = n_2 = 1, n_3 \geq 8$ with a cyclic permutation.
- Diller : construction of (almost all) quadratic maps fixing a cubic with one singular point.
- Blanc : construction of higher degree maps with a curve of fixed points.

Diller's construction

Let C be a cubic curve with one singularity
(a cubic with a cusp, a conic and its tangent, the lines joining at a single point)

Suppose $f = L \circ J$ such that

- All points of indeterminacy for both f and f^{-1} lie on C.
- $\overline{f(C \setminus \mathcal{I})} = C$

In this case, we say f properly fixes C.

$f|_{C_{reg}}$ together with the locations of points of indeterminacy and critical images determines $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$

Not very good explanation

- each column ℓ_i of L corresponds to a point of indeterminacy for f^{-1}
$$\ell_i = \lambda_i p_i$$
- since f fixes C, we have enough conditions to determine λ_i's.

Better explanation

- Any line ℓ passing through exactly one point e_i of indeterminacy is mapped to a line.
- $\#\ell \cap C = 3$ (counted with multiplicity)
- $\#\ell \cap \{x_i = 0\} = 1$. Thus $f(\ell)$ passes through p_i.
- three points of indeterminacy are not collinear.
- One can determine the image of any point $p \notin C$ using three lines $\overline{p\ e_i}$

Diller's construction

We need to determine the locations of points of indeterminacy.

- Specify orbit lengths n_1, n_2, n_3 and a permutation σ to guarantee that the resulting birational map lifts to an automorphism with positive entropy.
- Using the group law on Elliptic curves, determine the restriction map $f|_C$ and points of indeterminacy.

Diller's construction

Very explicit!!

$n_1 = n_2 = 1, n_3 = 8$ and a cyclic permutation $1 \to 2 \to 3 \to 1$
Suppose f properly fixes a cubic curve C with a cusp.

$$
\begin{aligned}
f &= L \circ J \quad \text{where} \\
L &= \begin{pmatrix} 0 & 0 & 1 \\ \alpha^9 - \alpha^6 - \alpha^4 + 1 & 0 & -\alpha^9 + \alpha^6 + \alpha^4 \\ 0 & \alpha^3 + \alpha^2 & -\alpha^3 - \alpha^2 + 1 \end{pmatrix}, \\
J[x1 : x2 : x3] &= [x2x3 : x1x3 : x1x2]
\end{aligned}
$$

$$
\text{where } \alpha^{10} + \alpha^9 - \alpha^7 - \alpha^6 - \alpha^5 - \alpha^4 - \alpha^3 + \alpha + 1 = 0
$$

$$
f|_C : \gamma(t) \mapsto \gamma(1/\alpha t + b), \quad C_{reg} = \{\gamma(t), t \in \mathbb{C}\}
$$

$$\chi(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$$

- $\chi(t)$ has exactly one real root λ outside the unit circle, exactly one real root $1/\lambda$ inside the unit circle. All other roots are non-real complex numbers of modulus 1.

$$\lambda \approx 1.1762$$

- Each root α of $\chi(t)$ determines a birational map f_α such that the multiplier of the restriction map is $1/\alpha$.
- $(t-1)\chi(t)$ is the characteristic polynomial of the induced action on cohomology. The dynamcial degree of $f = \lambda > 1$.
- there are two maps $f_\lambda, f_\lambda^{-1}$ with real coefficients.

$$\begin{cases} f_\lambda : \mathbf{P}^2(\mathbb{C}) \dashrightarrow \mathbf{P}^2(\mathbb{C}) \\ f_\lambda : \mathbf{P}^2(\mathbb{R}) \dashrightarrow \mathbf{P}^2(\mathbb{R}) \end{cases}$$

f_λ lifts to an automorphism $F : X \to X$ on a rational surface. We have two related dynamics.

$$\begin{cases} F : X(\mathbb{C}) \to X(\mathbb{C}) \\ F_{\mathbf{R}} : X(\mathbb{R}) \to X(\mathbb{R}) \end{cases}$$

- $h_{top}(F) = \log \lambda \approx \log 1.1762$
- $h_{top}(F_{\mathbf{R}}) \leq h_{top}(F)$
- $F_{\mathbf{R}}$ has maximal entropy if $h_{top}(F_{\mathbf{R}}) = h_{top}(F)$

Let C be a curve with one singularity.
For each orbit data (with few exceptions), there exists two real rational surface automorphisms properly fixing C.

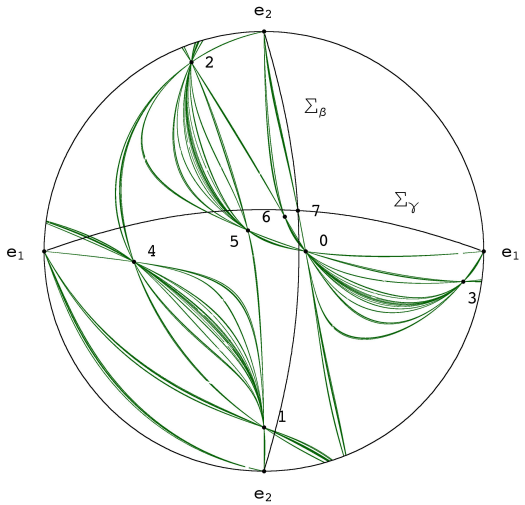


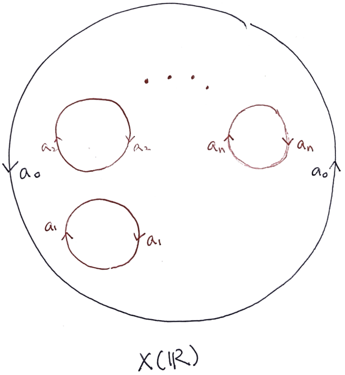
Figure: $F_{\mathbf{R}}$ associated with $1, 1, 8$ with cyclic permutation: The Image of a simple closed curve under 40 iterations.
Projected to $\mathbf{P}^2(\mathbb{R})$

Real Rational Surfaces

$X(\mathbb{R}) =$ a blow up of $\mathbf{P}^2(\mathbb{R})$ along a finite set of points

$$X(\mathbb{R}) = \mathbf{P}^2(\mathbb{R})\#\mathbf{P}^2(\mathbb{R})\#\cdots\#\mathbf{P}^2(\mathbb{R})$$

the connected sum of $n+1$ copies of $\mathbf{P}^2(\mathbb{R})$



Real Rational Surfaces

$X(\mathbb{R}) =$ a blow up of $\mathbf{P}^2(\mathbb{R})$ along a finite set of points

$X(\mathbb{R}) \setminus D =$ a disk with $n+1$ twisted handles attached to the boundary.

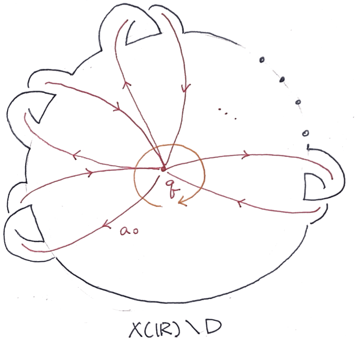


Figure: Each twisted handle is Möbius band

Real Rational Surfaces

$$\begin{aligned} X(\mathbb{R}) &= \text{a blow up of } \mathbf{P}^2(\mathbb{R}) \text{ along a set of } n \text{ points} \\ &= \underbrace{\mathbf{P}^2(\mathbb{R})\#\mathbf{P}^2(\mathbb{R})\#\cdots\#\mathbf{P}^2(\mathbb{R})}_{n+1 \text{ copies}} \end{aligned}$$

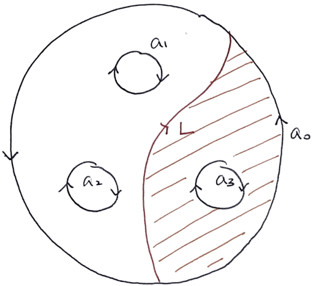
- $H_1(X(\mathbb{R})) = \langle a_0, a_1, \ldots, a_n | a_0^2 a_1^2 \cdots a_n^2 = 1, [a_i, a_j] = 1 \rangle$
  the finitely generated abelian group with one relation.
- $\pi_1(X(\mathbf{R}), q) = \langle a_0, a_1, \ldots, a_n | a_0^2 a_1^2 \cdots a_n^2 = 1 \rangle$
  the finitely generated NON-abelian group with one relation.
- $X(\mathbb{R})$ is a non-orientable surface!!

Entropy Estimation: Growth rate of homology classes.

$F_{\mathbf{R}} : X(\mathbb{R}) \to X(\mathbb{R})$ is a diffeomorphism associated with a birational map $f = L \circ J$ properly fixing a cusp cubic C

- The induced action $F_{\mathbf{R}*}$ on $H_1(X(\mathbb{R}))$ is a linear action.
- there are three different types of generators:
  - a_0 : the class of generic line.
  - a_i, $i \neq 0$: the class of an exceptional curve over a point of indeterminacy.
  - All other generators are mapped to $\pm$ another generator.
- Two main issues
  - the class of a curve with respect to the generators.
  - the orientation.

The class of an oriented curve



Using the boundary of a bounded region, we have

- the boundary of the shaded region ~ 0
- $L \sim a_0 + 2a_3$
- $L \sim -a_0 - 2a_1 - 2a_2$ using the unshaded region

Because of one relation

$$2a_0 + 2a_1 + 2a_2 + 2a_3 \sim 0 \;\Rightarrow\; a_0 + 2a_3 \sim -a_0 - 2a_1 - 2a_2$$

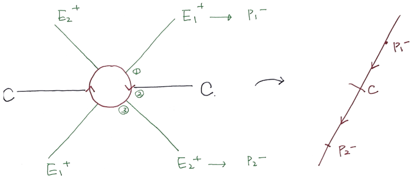
The Invariant cubic makes it easy

Every base point is between two fixed points in the invariant cubic.

The order of base points is determined by orbit lengths and the permutation.

The intersection between a curve L and the invariant cubic C will determine the bounded region with L as its boundary.

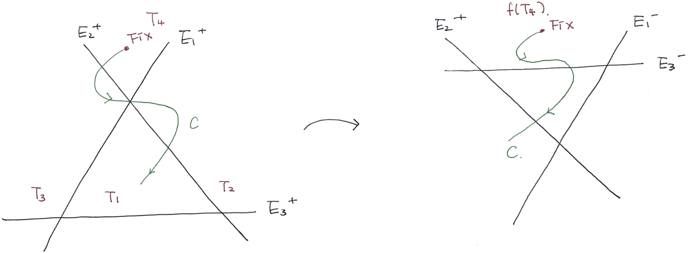
The orientation of the image of the exceptional curve over a point of indeterminacy



Orientation for images of other generators

All other cases, we only need to determine the orientation.

We only need to consider 4 open triangles determined by exceptional lines of f and f^{-1}



We have the induced action on homology classes

For all most all choices of orbit data $n_1, n_2, n_3, \in S_3$ (Some can not have a map fixing a cusp cubic), we have the induced action on homology classes

Theorem (Diller - K)

- $\chi(F_{\mathbf{R}*})$ *is reciprocal and* $\chi(F_{\mathbf{R}*}) = \chi(F_{\mathbf{R}*}^{-1})$
- *There are rational surface automrophisms* F *such that* $F_{\mathbf{R}}$ *has maximal entropy*
  - $1, 1, n$ *with a cyclic permutation*
  - $2, 2, n$ *with a cyclic permutation*
  - $2, 3, n$ *with the identity permutation*
  - $1, 4, n$ *with the transposition* $1 \leftrightarrow 2$
- *There is a complex rational surface automorphism such that all periodic cycles lie in the real locus.*

- There is a family of maps such that their real restriction $F_{\mathbf{R}}$ do not have maximal entropy.

$$\text{e.g.} \quad 3, 3, n \text{ with a cyclic permutation}$$

- There are maps such that $F_{\mathbf{R}*}^{k} = Id$ for some k.

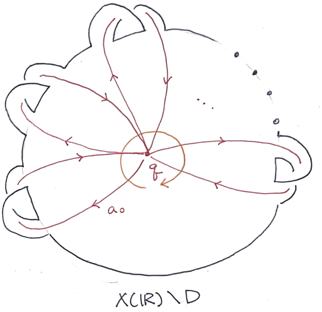
$$\text{e.g.} \quad \begin{array}{lcll} 1, 4, 8, \text{ with a cyclic permutation} & : & \text{period} & = 180 \\ 2, 3, 5, \text{ with a cyclic permutation} & : & \text{period} & = 84 \end{array}$$

This does not mean F_R has zero entropy.

We need better estimates

Entropy Estimation: Growth rate of homotopy classes.

Recall that we do have a natural choice for a set of generators for the fundamental group for $X(\mathbb{R})$

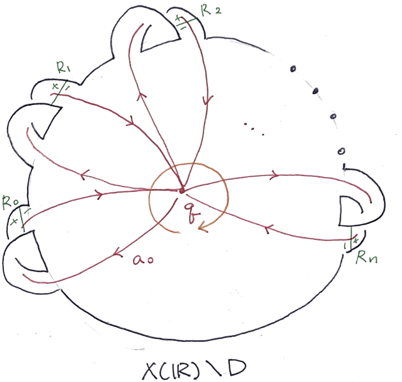


- We want to iterate the map. So let $q =$ a non-cusp fixed point (a saddle point) on the invariant cubic.
- We want to determine the image of each generator under the action $F_{\mathbf{R}*}$

Reading curves

Idea

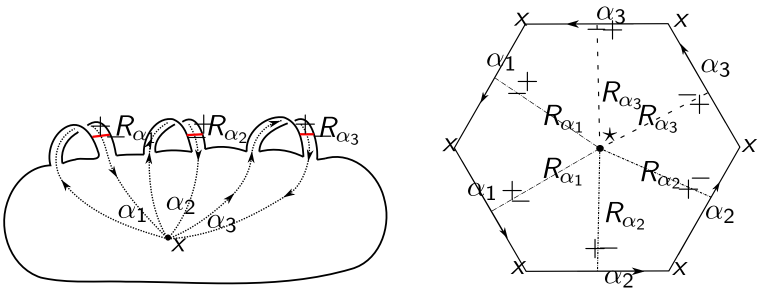
- For each generator α, find a curve X_α with $[X_\alpha] = \alpha$.
- *Calculate* the class $[F_{\mathbf{R}}(X_\alpha)]$ of the image curve
- With exceptional triangles and the invariant cubic, it is not hard to find $F_{\mathbf{R}}(X_\alpha)$



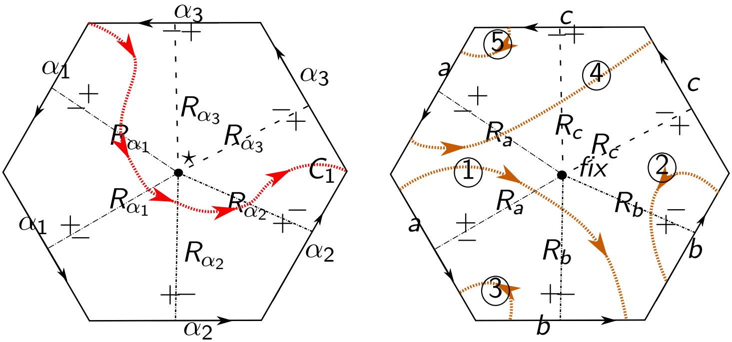
For each i, let R_{α_i} denote a line segment joining the two sides of the boundary of the handle traversed by the generator α_i. Once we put the removed open disk Δ back, we can extend each line segment R_{α_i} to a simple closed curve (which we continue to denote by R_{α_i}) with base point $\star \in \Delta$ such that

- R_{α_i} is a simple closed curve for all i,
- For each i, R_{α_i} intersects exactly one generator α_i, and
- $\{R_{\alpha_i}\}$ are pairwise disjoint on $X \setminus \{\star\}$.

The curves $\{R_{\alpha_i}\}$ are referred to as *reading curves* for the generators $\{\alpha_i\}$.



Generators and Reading curves on $X \equiv \mathbb{P}^2(\mathbf{R})\#\mathbb{P}^2(\mathbf{R})\#\mathbb{P}^2(\mathbf{R})$: The left-hand side is $X \setminus \Delta$ and the right-hand side is a polygon representation of X. In both figures, $\alpha_1, \alpha_2, \alpha_3$ are generators of $\pi_1(X, x)$ and $R_{\alpha_1}, R_{\alpha_2}, R_{\alpha_3}$ are corresponding reading curves.



A simple closed oriented curve C in $X \equiv \mathbb{P}^2(\mathbf{R})\#\mathbb{P}^2(\mathbf{R})\#\mathbb{P}^2(\mathbf{R})$:
On the left-hand side, the π_1-class of C_1 is given by $\alpha_1^2\alpha_2^2 = \alpha_3^{-2}$.
And on the right-hand side the π_1-class of C_2 is given by $\alpha_1^{-1}\alpha_3^{-1}$.

With E. Klassen, we compute the induced π_1 action for real diffeomorphisms associated with birational maps fixing a cusp cubic with orbit data n_1, n_2, n_3 and a cyclic permutation.

Eg. For $n_1 = n_2 = 1, n_3 = 8$

$$
\begin{aligned}
f_{\mathbf{R}*}^{-1} \;:\; e &\mapsto e a_8^2 a_7^2 a_6^2 a_5^2 c_1 b_1 a_2^2 a_1 e \\
a_8 &\mapsto e^{-1} a_1^{-1} a_2^{-2} b_1^{-1} \\
a_7 &\mapsto b_1 a_2^2 a_1 e a_8^{-1} e^{-1} a_1^{-1} a_2^{-2} b_1^{-1} \\
a_6 &\mapsto b_1 a_2^2 a_1 e a_8^2 a_7^{-1} a_8^{-2} e^{-1} a_1^{-1} a_2^{-2} b_1^{-1} \\
a_5 &\mapsto b_1 a_2^2 a_1 e a_8^2 a_7^2 a_6^{-1} a_7^{-2} a_8^{-2} e^{-1} a_1^{-1} a_2^{-2} b_1^{-1} \\
a_4 &\mapsto b_1^{-1} a_3^{-2} a_4^{-2} c_1^{-1} a_5 c_1 a_4^2 a_3^2 b_1 \\
a_3 &\mapsto b_1^{-1} a_3^{-2} a_4^{-1} a_3^2 b_1 \\
a_2 &\mapsto e a_8^2 a_7^2 a_6^2 a_5^2 c_1 a_3 c_1^{-1} a_5^{-2} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
a_1 &\mapsto e a_8^2 a_7^2 a_6^2 a_5^2 c_1 b_1 a_2^{-1} b_1^{-1} c_1^{-1} a_5^{-2} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
b_1 &\mapsto b_1^{-1} a_3^{-2} c_1^{-1} a_5^{-2} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
c_1 &\mapsto b_1 a_2^2 a_1 e a_8^2 a_7^2 a_6^2 c_1 a_4^2 a_3^2 b_1.
\end{aligned}
$$

$$\pi_1(X(\mathbb{R})) = \langle e, a_i, b_1, c_1 | e^2 a_8^2 a_7^2 a_6^2 a_5^2 c_1^2 a_4^2 a_3^2 b_1^2 a_2^2 a_1^2 = 1 \rangle$$

- We want to Calculate the growth rate.

$$\rho(f_{\mathbf{R}*}|_{\pi_1(X(\mathbf{R})}) := \sup_{g \in G}\{\limsup_{n\to\infty}(\ell_G(f^n_{\mathbf{R}*}g))^{1/n}\}$$

  where G is a set of generators, and $\ell_G(w)$ is the minimal length among all words representing w with respect to G.

- $\pi(X(\mathbb{R}))$ is a non-abelian group with a relator.

- We want to ~~Calculate~~ Estimate the growth rate.

$$\rho(f_{\mathbf{R}*}|_{\pi_1(X(\mathbf{R}))}) := \sup_{g \in G}\{\limsup_{n\to\infty}(\ell_G(f^n_{\mathbf{R}*}g))^{1/n}\}$$

  where G is a set of generators, and $\ell_G(w)$ is the minimal length among all words representing w with respect to G.

- $\pi(X(\mathbb{R}))$ is a non-abelian group with a relator.

- For $\alpha \in \pi_1(X(\mathbb{R}))$, the minimum length $\ell_G(\alpha)$ is obtained by removing more than half-relators.

Theorem (E. Klassen -K)

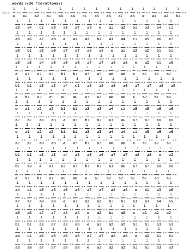
There are real quadratic rational surface automorphisms with maximal entropy such that the growth rate of homology classes is strictly smller than the growth rate of homotopy classes

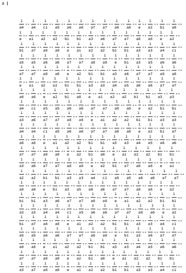
$$\begin{aligned} e.g. \quad & n_1 = 1, n_2 = 3, n_3 = 9 \quad \textit{with a cylcic permutation} \\ & n_1 = 1, n_2 = 4, n_3 = 8 \quad \textit{with a cylcic permutation} \\ & n_1 = 1, n_2 = 4, n_3 = 5 \quad \textit{with a cylcic permutation} \\ & n_1 = 1, n_2 = 5, n_3 = 6 \quad \textit{with a cylcic permutation} \end{aligned}$$

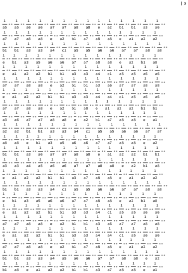
The exponential homology growth rates for the first two cases above are zero.

How to estimate the growth rate: $1, 1, 8$ cyclic case

Fist, we examined the iterations under $f_{\mathbf{R}*}$







Obviously, you see the clear pattern, right?

There are ten reduced elements

$$\Gamma = \{\gamma_i \in \pi_1(X(\mathbb{R})), 1 \leq i \leq 10\}$$

and a subset A of the set of ordered pairs

$$A \subset \{(i,j)|1 \leq i,j, \leq 10\}$$

such that

- There are no relations between γ_i's
- We say. γ is A-admissible if $\gamma = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_n}$ with $(i_j, i_{j+1}) \in A, j = 1, \ldots, n \pmod n$
- If γ is A-admissible then $f_{\mathbf{R}*}\gamma$ is also admissible up to cyclic permutation.

The set Γ_A of admissible cyclic words is invariant under $f_{\mathbf{R}*}\gamma$.

The action on the admissible set Γ_A

- There is a subset $\Gamma_1 \subset \Gamma$ and a finite set K of elements in $\pi_1(X(\mathbb{R})) \setminus \Gamma$ such that

$$f_{\mathbf{R}*}\gamma_i = \kappa_{i_l}\gamma'\kappa_{i_r} \ \dot{} where \gamma'_i \in \Gamma_A, \kappa_{i_l}, \kappa_{i_r} \in K$$

- For $(i,j) \in A$, $\kappa_{i_r}\kappa_{i_l} = \gamma_s$ and

$$\gamma'_i\gamma_s\gamma'_j \text{ is } A\text{-admissible.}$$

$$\gamma_1 = a_2^{-1} b_1^{-1} a_8^{-1} e^{-1}, \qquad \gamma_2 = a_2^{-1} b_1^{-1} a_7^{-1} a_8^{-2} e^{-1}$$

$$\gamma_3 = b_1^{-1} a_3^{-1} a_6^{-1} a_7^{-2} a_8^{-2} e^{-1}$$

$$\gamma_4 = a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-1} a_7^{-1} a_8^{-2} e^{-1}$$

$$\gamma_5 = a_2^{-1} b_1^{-2} a_3^{-1} a_6^{-1} a_7^{-2} a_8^{-2} e^{-1}$$

$$\gamma_6 = b_1^{-1} a_3^{-1} a_5^{-1} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1}$$

$$\gamma_7 = a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-1} a_6^{-1} a_7^{-2} a_8^{-2} e^{-1}$$

$$\gamma_8 = a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-2} a_4^{-1} a_5^{-1} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1}$$

$$\gamma_9 = a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-2} a_4^{-2} c_1^{-1} . a_5^{-1} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1}$$

$$\gamma_{10} = a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-2} a_4^{-1} c_1^{-1} a_5^{-2} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1}$$

$$\zeta_1 = a_2 a_1, \qquad \zeta_2 = b_1 a_2^2 a_1$$

$$\mu_1 = a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-1} a_5^{-1} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1}$$

$$A = \{(1,4),(1,7),(1,10),(2,8),(2,10),(3,2),\\ (3,8),(3,9),(4,10),(5,2),(5,8),(5,9),(6,1),(6,5),\\ (7,2),(8,1),(9,1),(9,2),(10,3),(10,5),(10,6)\}.$$

$$\begin{aligned}
f_{\mathbf{R}*}^{-2}\gamma_1 &= (e^{-1}a_1^{-1})\gamma_{10}\zeta_2(a_1 e), & f_{\mathbf{R}*}^{-2}\gamma_2 &= (e^{-1}a_1^{-1})\gamma_{10}\zeta_1(a_1 e)\\
f_{\mathbf{R}*}^{-2}\gamma_3 &= (e^{-1}a_1^{-1})\gamma_9\gamma_1(a_1 e), & f_{\mathbf{R}*}^{-2}\gamma_4 &= (e^{-1}a_1^{-1})\mu_1\zeta_1(a_1 e)\\
f_{\mathbf{R}*}^{-2}\gamma_5 &= (e^{-1}a_1^{-1})\gamma_8\gamma_1(a_1 e), & f_{\mathbf{R}*}^{-2}\gamma_6 &= (e^{-1}a_1^{-1})\gamma_9\gamma_2(a_1 e)\\
f_{\mathbf{R}*}^{-2}\gamma_7 &= (e^{-1}a_1^{-1})\mu_1\gamma_1(a_1 e), & f_{\mathbf{R}*}^{-2}\gamma_8 &= (e^{-1}a_1^{-1})\gamma_7\gamma_2(a_1 e)\\
f_{\mathbf{R}*}^{-2}\gamma_9 &= (e^{-1}a_1^{-1})\gamma_4(a_1 e), & f_{\mathbf{R}*}^{-2}\gamma_{10} &= (e^{-1}a_1^{-1})\gamma_7(a_1 e)
\end{aligned}$$

$$\gamma_3 = \zeta_2\gamma_7, \quad \gamma_5 = \zeta_1\gamma_7, \quad \text{and} \quad \gamma_6 = \zeta_2\mu_1$$

Since there is no relation, We can get the length growth by counting number of γ_i's

$$V = \mathbb{R}^{10} \quad \text{with a basis } \{\gamma_1, \dots \gamma_{10}\}$$
$$W = \mathbb{R}^{|\Gamma_1|+|K|} \quad \text{with a basis } \Gamma_1 \cup K$$

Theorem (Klassen-K)

There are two linear maps $S, T : V \to W$ such that

$$T \circ f_{\mathbf{R}*}|_{\Gamma_A} = S$$

and there is a unique vector $v \in V$ such that $Sv = \lambda Tv$ where λ is the dynamical degree of f

We observed that the same phenomenon occurs in other orbit data.

$$T \circ f_{\mathbf{R}*}|_{\Gamma_A} = S$$

where $S, T : V \to W$ are linear.
Is $f_{\mathbf{R}*}|_{\Gamma_A}$ (almost) linear?

There are ten A-admissible words $S = \{s_1, \ldots, s_{10}\}$ where the action. $f_{\mathbf{R}*}|_{SP^+S}$ on the positive span of S is "Linear"

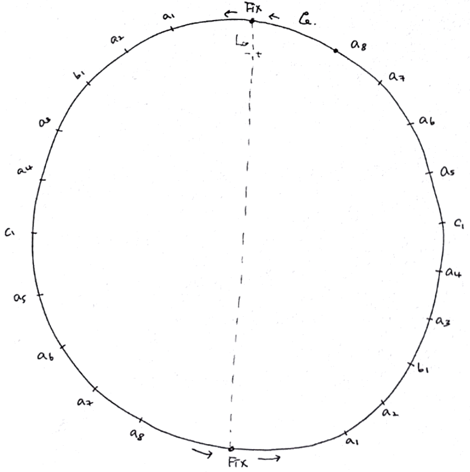
$$s_1 = \gamma_1\gamma_{10}\gamma_5 \cdots, \gamma_8$$

$$\vdots$$

$$\begin{aligned}
f_{\mathbf{R}*}|_{SP^+S} : s_1 &\mapsto s_1 + s_2 + s_3 + s_4 + s_5 \\
s_2 &\mapsto s_1 + s_6 + s_7 \\
s_3 &\mapsto s_1 + s_2 + s_8 + s_3 + s_4 + s_5 \\
s_4 &\mapsto s_1 + s_2 + s_4 + s_{10} + s_6 + s_9 \\
s_5 &\mapsto s_1 + s_4 + s_5 + s_7 \\
s_6 &\mapsto s_1 + s_6 + s_9 + s_5 + s_7 \\
s_7 &\mapsto s_1 + s_2 + s_8 + s_9 \\
s_8 &\mapsto s_1 + s_2 + s_4 + s_{10} + s_6 + s_9 + s_5 + s_7 \\
s_9 &\mapsto s_1 + s_2 + s_8 + s_3 + s_4 + s_{10} + s_6 + s_9 \\
s_{10} &\mapsto s_1 + s_4 + s_5
\end{aligned}$$

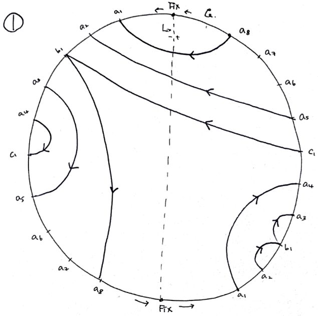
Sequence of Admissible Words

Since the cubic is invariant and all base loci lie between two fixed points on the cubic, the $X(\mathbb{R})$ can be drawn as following:

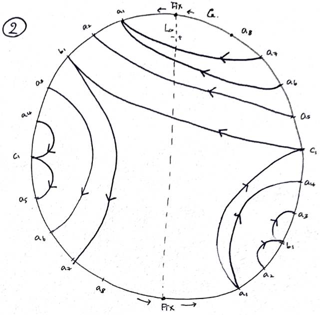


Sequence of Admissible Words

Starting with an admissible word γ, under $f^2_{\mathbf{R}*}$ we see



$\gamma_1 \gamma_{10} \gamma_6$

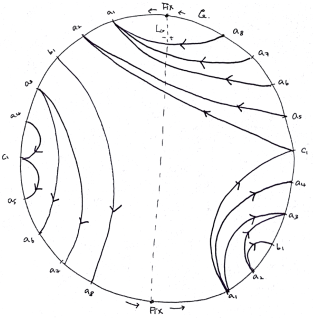


$\gamma_{10} \gamma_3 \gamma_9 \gamma_2$

Sequence of Admissible Words

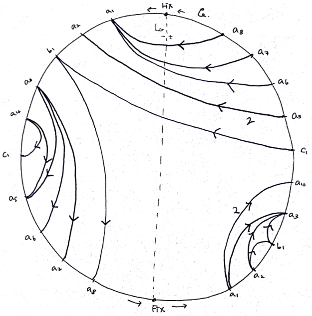
Starting with an admissible word γ, under $f^2_{\mathbf{R}*}$ we see

③



$\gamma_1 \gamma_4 \gamma_{10} \gamma_5 \gamma_9$

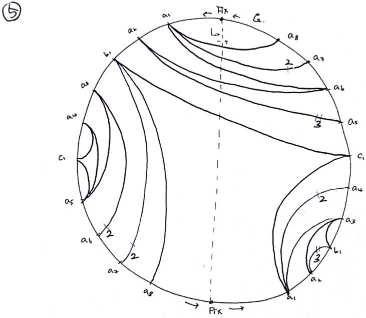
④



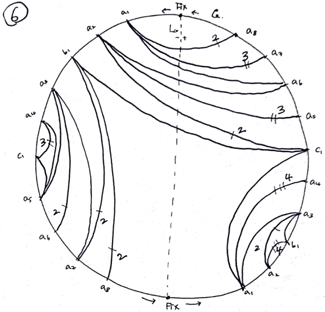
$\gamma_1 \gamma_4 \gamma_{10} \gamma_6 \gamma_5 \gamma_8$

Sequence of Admissible Words

Starting with an admissible word γ, under $f^2_{\mathbf{R}*}$ we see



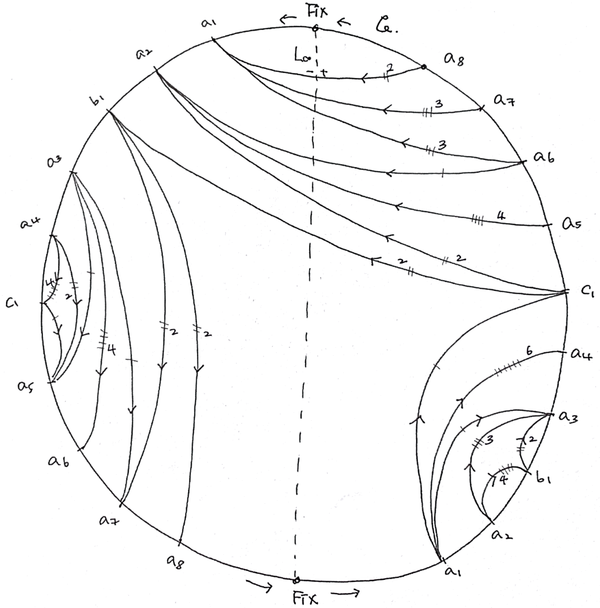
$\gamma_1\gamma_7\gamma_2\gamma_{10}\gamma_6\gamma_5\gamma_9\gamma_2\gamma_8$



$\gamma_1\gamma_4\gamma_{10}\gamma_5\gamma_2\gamma_{10}\gamma_6\gamma_1\gamma_{10}\gamma_5$

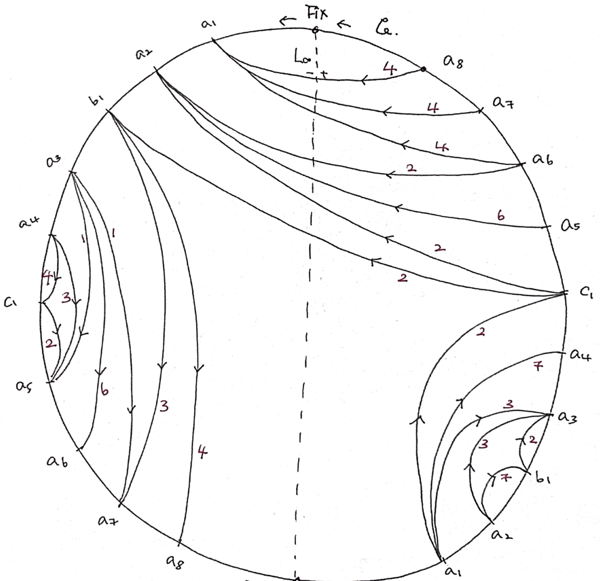
Sequence of Admissible Words

Starting with an admissible word γ, under $f^2_{\mathbf{R}*}$ we see

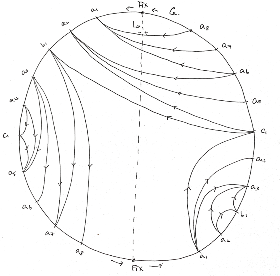


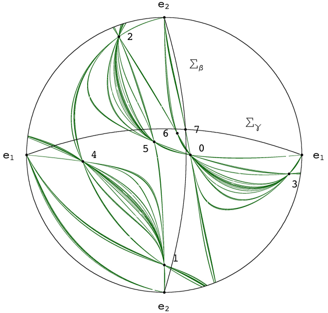
Sequence of Admissible Words

Starting with an admissible word γ, under $f^2_{\mathbf{R}*}$ we see



It "seems" that with any choice of an initial word, we get the same picture with increasing weight (the number of arcs).





- [Kitchens-Roeder] Is this a Plykin attractor?
  - We know this grows exponentially.
  - There is a repelling fixed point whose basin has full Area. [Bedford-K]
- Are we seeing a pseudo-anosov map on non-orientalbe surface with genus 11 and no puncture?
  - Is this a Train Track (a Ribbon Map)?
  - Can we use Nielson-Thurston Classification?

Thank you!!

감사합니다!