Dynamics of Complex Hénon Maps

Eric Bedford

Stony Brook U.

Complex Hénon maps. Generalization of the quadratic family $x \to x^2 + c$

$$H(x,y) = (x^2 + c - ay, x) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$H^{-1}(x,y) = \left(y, \frac{y^2 + c - x}{a} \right)$$

Or more generally

$$H(x,y) = (p(x) - ay, x)$$ p polynomial of degree $d \geq 2$. $a \neq 0$

In analogy with 1D :

$$K^+ = \{ (x,y) : \{ H^n(x,y), n \geq 0 \} \text{ bounded} \}$$

$$K^- = \text{- - - - - } H^{-n} \text{ - - - - - - - - - - -}$$

$$K = K^+ \cap K^-$$

Rate of escape to infinity

$$G^+(x,y) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \| H^n(x,y) \|$$

G^- 〃 〃 〃 〃 〃 〃 〃 H^{-n} 〃 〃 〃

<u>Thm</u> (Hubbard) $G^\pm : \mathbb{C}^2 \longrightarrow \mathbb{R}_+$ is continuous

1. $G^+ \circ H = d \cdot G^+$ $\qquad G^- \circ H = \frac{1}{d} \cdot G^-$
2. $\{G^\pm = 0\} = K^\pm$
3. $G^\pm$ is pluri-harmonic on $U^\pm := \mathbb{C}^2 - K^\pm$

Hubbard's "Invitation": study H in terms of the fibration $G^+ : U^+ \longrightarrow (0,\infty)$

U^+ is an interesting set from point of view of complex analysis. My "previous" talk

Digression: Can we extend H to a compact, complex manifold?

Extend to projective space.

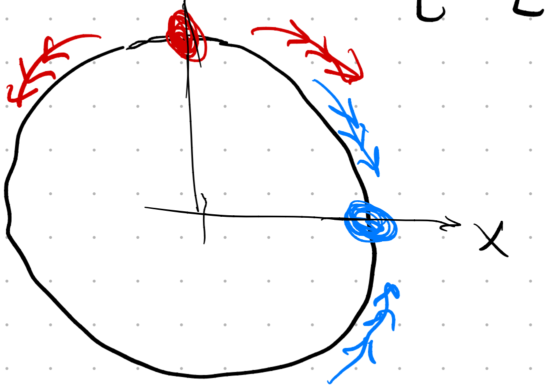
$$\mathbb{C}^2 \ni (x,y) \longrightarrow [x:y:1] \in \mathbb{P}^2$$

$H(x,y) = (x^2+c-ay, x)$ becomes

$$H[x:y:z] = [x^2+cz^2-ayz : xz : z^2]$$

Line at ∞, $L_\infty = \{z=0\}$ invariant under H, H^{-1}

$U^+ =$ forward basin of $[1:0:0] \in L_\infty$



$H(L_\infty - ●) = ●$

$H(●) = L_\infty$

dynamical degree is

$$\delta(f) := \lim_{n \to \infty} \left(\deg(f^n)\right)^{1/n}$$

Thm (Diller-Favre)

$\delta(f)$ is algebraic, birational invariant.

If $f \overset{\text{birat}}{\simeq}$ automorphism of compact surface, and if $\delta(f) > 1$, then f is irrational.

But: $\delta(H) = \deg P \geq 2$ integer.

Thus:

No Hénon map H is birationally conj. to compact surface automorphism.

Thm Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be birational.
Then

f is birationally conjugate to a poly aut $\curvearrowright \mathbb{C}^2$

$\Updownarrow$

$\exists$ Line $L \subset \mathbb{P}^2$ invariant under f, f^{-1}

$L = L_\infty$ is polar locus for $G^\pm$

Cor: Having global potential $G^\pm$ possible only in Hénon case

Compact surface automorphisms have invariant currents $T^{+/-}$. Theory is almost same as for Hénon case.

Here we focus on properties related to G^+.

Goal: Apply complex analytic methods to obtain dynamical results. At the same time the dynamical context should lead to interesting situations for complex analysis.

N. Sibony and I responded to Hubbard's invitation from the point of view of psh functions as potentials

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Def u is **psh** if it is subharmonic in any coordinate system, i.e. X biholo $\Rightarrow u \circ X$ s.h.

Language of Oka: such functions are **pseudoconvex**.

If V is an electric potential (voltage), then

$$\Delta V = 4 \frac{\partial^2}{\partial z \partial \bar{z}} V$$ is the charge distribution

Higher dimension: replace Δ by $dd^c := \sum \frac{\partial^2}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$

not conformally invariant in $\mathbb{C}^2$ (even for linear $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$))

biholo. invariant

Invariant currents:

$$\mu^+ = dd^c G^+$$

$$\mu^- = dd^c G^-$$

$$H^* \mu^+ = d \cdot \mu^+ \qquad H^* \mu^- = \frac{1}{d} \cdot \mu^-$$

What is the structure of $\mu^{\pm}$?

If $\mathcal{D} \subset \mathbb{C}^2$ is a complex disk, $dd^c\big|_{\mathcal{D}}$ = "intrinsic Laplacian on $\mathcal{D}$"

slice measure on $\mathcal{D}$

$$\mu^+\big|_{\mathcal{D}} := \left(dd^c\big|_{\mathcal{D}}\right)\left(G^+\big|_{\mathcal{D}}\right)$$

<u>Prop.</u> μ^+ is (defined by) the family of slice measures $\mu^+\big|_{\mathcal{D}}$.

(Earlier) Pluri-Potential Theory of a compact $K \subset \mathbb{C}^2$

psh Green function

$$G_K = \sup \left\{ u : \begin{array}{l} u \text{ psh}, \quad u \leq 0 \text{ on } K \\ u \leq \log^+ \|z\| + C \end{array} \right\}$$

Pluri-complex equilibrium measure

$$\mu_K := (dd^c G_K)^2 \qquad \text{(B.A. Taylor)}$$

Thm (B-Taylor) $\operatorname{supp} \mu_K = \partial_s K$ Shilov boundary

(= smallest compact $E \subset K$ such that

$$\max_E |p| = \max_K |p| \quad \text{for}$$

all polynomials p)

Potential theory in case of Hénon maps

<u>Thm</u>: (B-S, book) $G_K = \max(G^+, G^-)$

$$\mu_K = \mu^+ \wedge \mu^-$$

μ_K is H-invariant measure.

Fatou sets (forward/backward)

$F^+ :=$ open set where $\{H^n, n \ge 0\}$ is normal (equicontinuous)

$F^- :=$ " " " " " " H^{-n} " " " " " "

Julia sets

$$J^\pm := \mathbb{C}^2 - F^\pm \underset{\substack{\uparrow \\ \text{thm}}}{=} \partial K^\pm = \partial U^\pm \underset{\text{Thm}}{=} \operatorname{supp} \mu^\pm$$

$$J := J^+ \cap J^-$$

$$K := K^+ \cap K^-$$

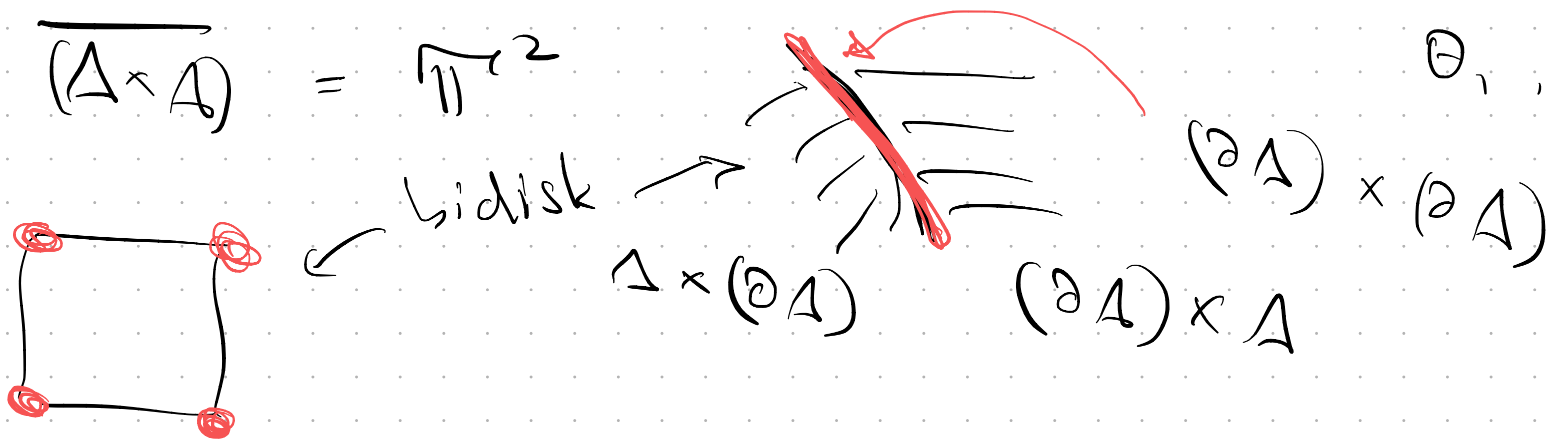
Shilov boundary - examples

(1a) $\mathcal{D}_1 \subset \mathbb{C}^2$ is a complex disk. $\partial_s \overline{\mathcal{D}_1} = \gamma$

(1b) $\mathcal{D}_2 \subset \mathbb{R}^2 \subset \mathbb{C}^2 \Rightarrow \partial_s \overline{\mathcal{D}_2} = \overline{\mathcal{D}_2}$

(2) $\mathbb{T}^2 = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1 \} = \{ (e^{i\theta_1}, e^{i\theta_2}) : \theta_1, \theta_2 \in \mathbb{R} \}$

$\partial_s \overline{(\Delta \times \Delta)} = \mathbb{T}^2$



(3) <u>Thm</u> (Joericke) There exist compact sets $S \subset \mathbb{C}^2$, $\text{int}(S) \neq \emptyset$, such that Shilov bdry $\partial_s S$ is a cantor set: compact, perfect, totally disconnected.

Another Julia set (1-D case: $J = \partial K$)

$J^* := \partial_s K = \operatorname{supp} \mu_K$

Clearly $J^* \subset J$

Thm (B-Lyubich-Smillie IV)

1. $J^* = \text{closure}\{\text{saddle (periodic) points}\}$

2. If p, q saddles

$$J^* = \text{closure}\left(W^s(p) \cap W^u(q)\right)$$

Question: $J = J^*$??

are both Julia sets the same.

Cases where $J = J^*$

not so hard:

Thm (B-Smillie III) If K is totally disconnected, then $J = J^*$.

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compact set S is hyperbolic if $\exists$ invariant splitting $T_x\mathbb{C}^2 = E^u_x \oplus E^s_x$ $\forall x \in S$, and DH is uniformly expanding/contracting on $E^{u/s}_x$

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much deeper and more recently:

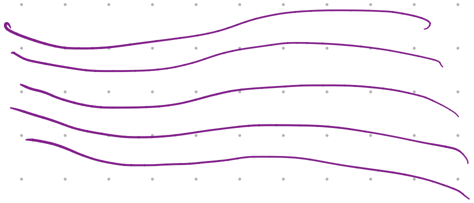
Thm (Dujardin) If J^* is a hyperbolic set, then $J = J^*$. (and we say H is hyperbolic)

Stable set of a point x:

$$W^s(x) = \{ y : \lim_{n \to +\infty} \operatorname{dist}(f^n x, f^n y) = 0 \}$$

Stable Manifold Theorem: If X is a hyperbolic set, then

$$\mathcal{W}^s = \bigcup_{x \in X} W^s(x)$$



is a Riemann surface lamination: locally homeo. to a product $S \times \Delta$

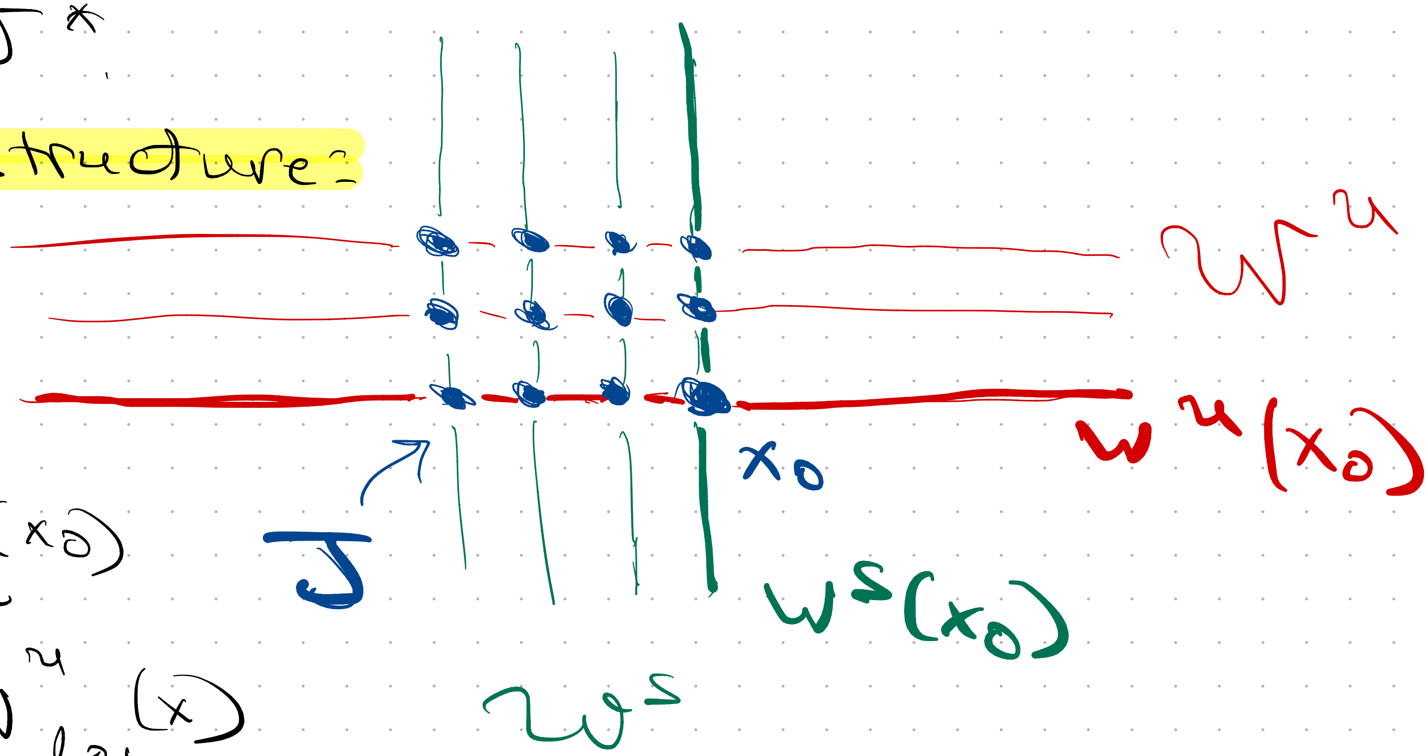
compact × disk

Hénon map H is hyperbolic if J is a hyperbolic set.

Thm (B-Smillie I) If H is hyperbolic, then

1. J has local product structure.
2. μ^+ slices measures measure $\mathcal{W}^{s/u}$
3. $J = J^*$.

Local prod structure:

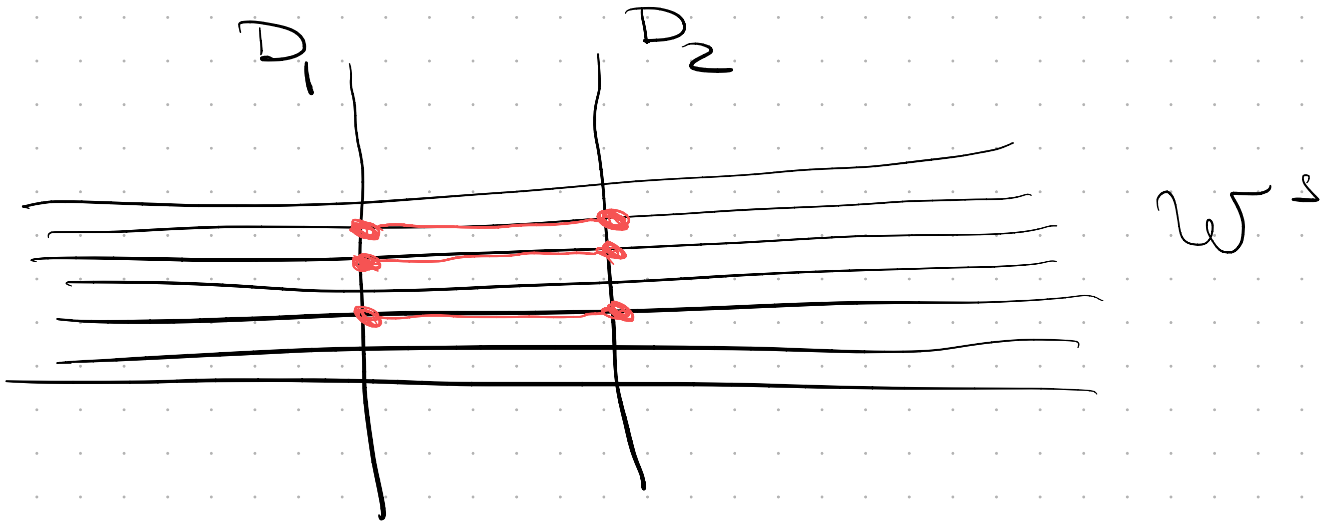


$$J^s_{loc} := J \cap W^s_{loc}(x_0)$$

$$J^u_{loc} := J \cap W^u_{loc}(x)$$

$$J \cap (\text{nbhd of } x_0) \cong J^s_{loc} \times J^u_{loc}$$

slice measures measure $\mathcal{W}^s$



holonomy map $\chi : D_1 \cap \mathcal{W}^s \longrightarrow D_2 \cap \mathcal{W}^s$

$$\mu^+|_{D_2} = \chi_*(\mu^+|_{D_1})$$

Currents

k-dim currents are the dual space to the test k-forms (smooth k-forms w. compact supp.)

M = oriented real surface $\subset \mathbb{C}^2$, locally finite area

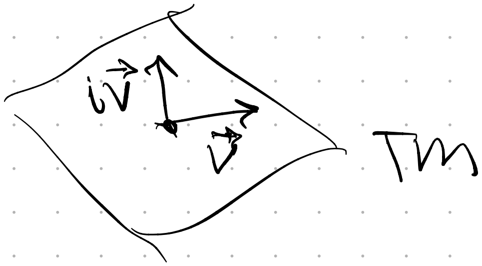
$[M]$ = current of integration

$$\langle [M], \varphi \rangle = \int_M \varphi$$

Complex magic

1. canonical orientation

$\vec{v} \wedge i\vec{v}$ = orientation 2-vector



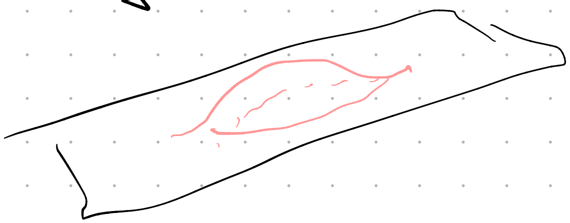
2. positivity   Example $x = ky^2$, $k \to \infty$

real case

complex case

3. complex surfaces locally minimize area

any bump will increase area



4. Wirtinger Formula

standard Kähler form = standard symplectic form

$$\beta = \frac{i}{2}\left[dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2\right] = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

$$\text{Area}(M) = \int_M \beta$$

Conversely: if $\beta|_M$ = Euclidean area form on M

then M is complex

5. (very weak) topology

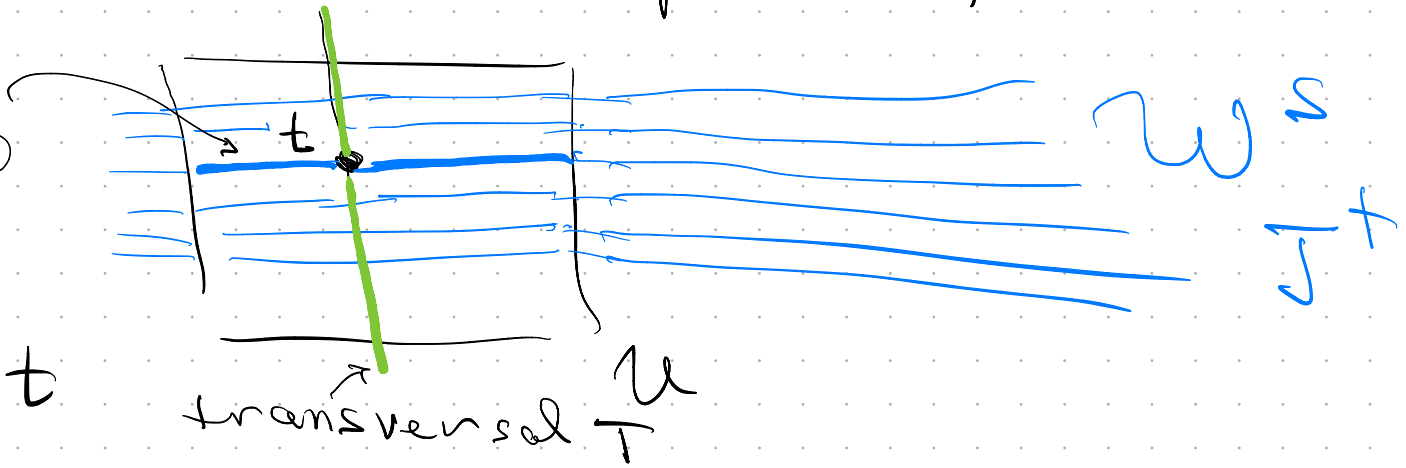
$$[M_j] \to [M] \text{ if}$$

$$\int_{M_j} \varphi \longrightarrow \int_M \varphi \quad \forall \varphi$$

Our interest: $M = W^s(p)$

Thm (B–Smillie I) If H is hyperbolic, then locally

m_t = component of stable manifold containing t



$$\mu^+|_U = \int_{t\in T} [m_t] \left(\mu^+|_T\right)(t)$$

μ^+ represents W^s as a current

- G^+ continuous $\Rightarrow$ J^+ "thick" $\Rightarrow$ no isolated disks
- $\mu^+|_U$ is reducible.

For non-hyperbolic maps, $\mathcal{W}^s$ is not a lamination (later), but still we have:

Thm (B-Lyubich-Smillie IV) There are uniformly laminar currents S_j, $j=1,2,3\dots$ on (small) open sets U_j such that

$$\mu^+ = \sum_{j=1}^{\infty} S_j .$$

In general:

Thm (B-Smillie II) Let p = saddle pt

D^s_p = stable disk about p.

$$\frac{1}{d^n}\left[H^{-n}(D^s_p)\right] \longrightarrow c\,\mu^+, \qquad c>0$$

$$c = \text{mass}\left(\mu^- \Big|_{D^s_p}\right)$$

Cor $W^s(p) = \bigcup_{n=1}^{\infty} H^{-n}(D^s_p)$ is dense in J^+,

Rmk analogous result for W^u

Irreducibility / Ergodicity of μ^+

Thm (Fornaess - Sibony) If T is a positive, closed current with $\operatorname{supp}(T) \subset K^+$, then

$$T = c\,\mu^+$$

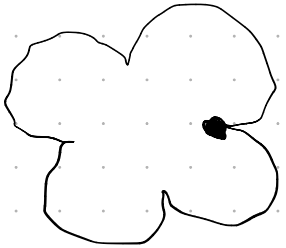
The theory of $\mu^+, \mu^-;$ $\mu = \mu^+ \wedge \mu$ works in almost the same way for compact surface automorphisms with $\delta(f) > 1$.

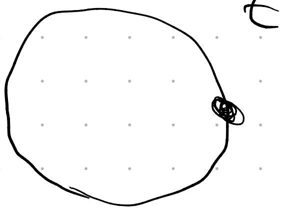
Cantat, Dujardin

Hyperbolicity not preserved under J-conjugacy

cauliflower map   conjugate on J to

$z \longrightarrow z^2$





For an example:

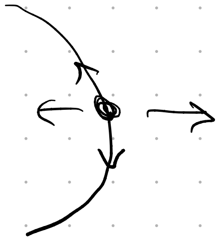
Thm (Radu-Tanase) ∃ semi-parabolic H (not hyperbolic) which is conjugate on J to hyperbolic $H = (x^2 - \varepsilon y, x)$

In this case, $J \underset{\text{homeo}}{\cong}$ Real solenoid.

$$H|_J \cong \text{angle doubling}$$

But if conjugate in a non-invariant "germ" about J:

In tiny neighborhood of fixed point:



<u>Thm</u> (B-Dujardin). Let H_1, H_2 be Henon maps. Suppose that there is a continuous map $\phi : \mathcal{U} \xrightarrow{\text{open}} \mathbb{C}^2$, $\mathcal{U} \supseteq J_{H_1}$, such that

$$\phi \circ H_1 = H_2 \circ \phi \quad \text{wherever defined.}$$

Then H_1 hyperbolic $\Longrightarrow$ H_2 hyperbolic

Is there some good / useful way of checking for hyperbolicity ??

"You can observe a lot just by watching."

Y. Berra

Can you observe hyperbolicity?

Is hyperbolicity revealed by geometry?

Ingredients for hyperbolicity

● spectral splitting / transversality
of E^s, E^u

●● uniform expansion/contraction

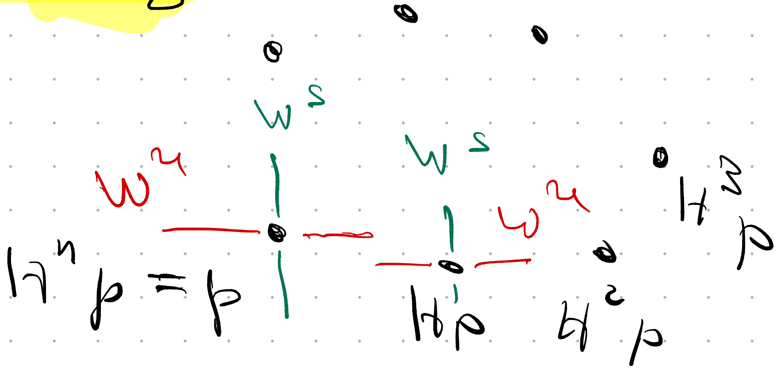
Possible "weakenings" of hyperbolicity

keep ● ⇝ partial hyperbolicity

keep ●● ⇝ quasi hyperbolicity.

$\mathcal{S} = \{$ saddle (periodic) points $\}$ $\longleftrightarrow$ DH^n_p expands / contracts

But we want uniform expansion on $\mathcal{S}$ at each step.



uniformization

$$\psi_p : \mathbb{C} \longrightarrow W^u(p)$$

unique modulo $\xi \longrightarrow \alpha \xi + \beta$

normalize ψ_p:

- $\psi_p(0) = p$
- $\max\limits_{|\xi| \le 1} G^+(\psi_p(\xi)) = 1$

ψ_p unique up to rotation $\xi \to e^{i\theta} \xi$

$$\Psi^u = \{ \text{normalized } \psi_p : p \in \mathcal{S} \}$$

For $p \in S^1$, define metric on $v \in E^u_p$

$$\|v\|^\#_p := \frac{|v|_{euclid}}{|\psi_p'(0)|_{euclid}} \qquad \psi_p \in \Psi^u_p$$

$\exists \lambda_p$: $H(\psi_p(\xi)) = \psi_{Hp}(\lambda_p \xi)$

- $\lambda_p = \|DH_p\|^\#_p$
- $G^+ \circ H = d \cdot G^+ \Rightarrow \lambda_p > 1$

H is quasi-expanding if Ψ^u is a normal family of entire mappings $\mathbb{C} \longrightarrow \mathbb{C}^2$

$\hat{\Psi}^u = \{$normal limits $\psi : \mathbb{C} \longrightarrow \mathbb{C}^2\}$

Thm (B-Smillie $\underline{\text{VIII}}$) Quasi expansion is equivalent to several things, including:

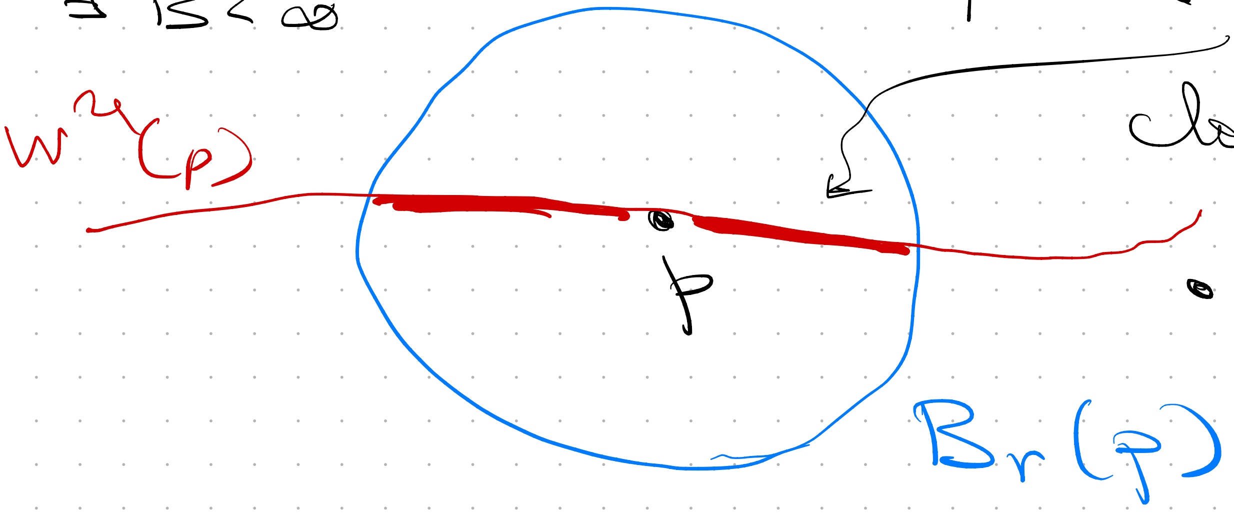
1. $\exists \kappa > 1$ such that $\lambda_p > \kappa \quad \forall p \in S$

   uniform expansion of $\| \ \|^{\#}$

2. Locally proper, bounded area

$\exists r > 0$: $\forall p \in S \quad \forall \psi_p$
$\exists B < \infty$

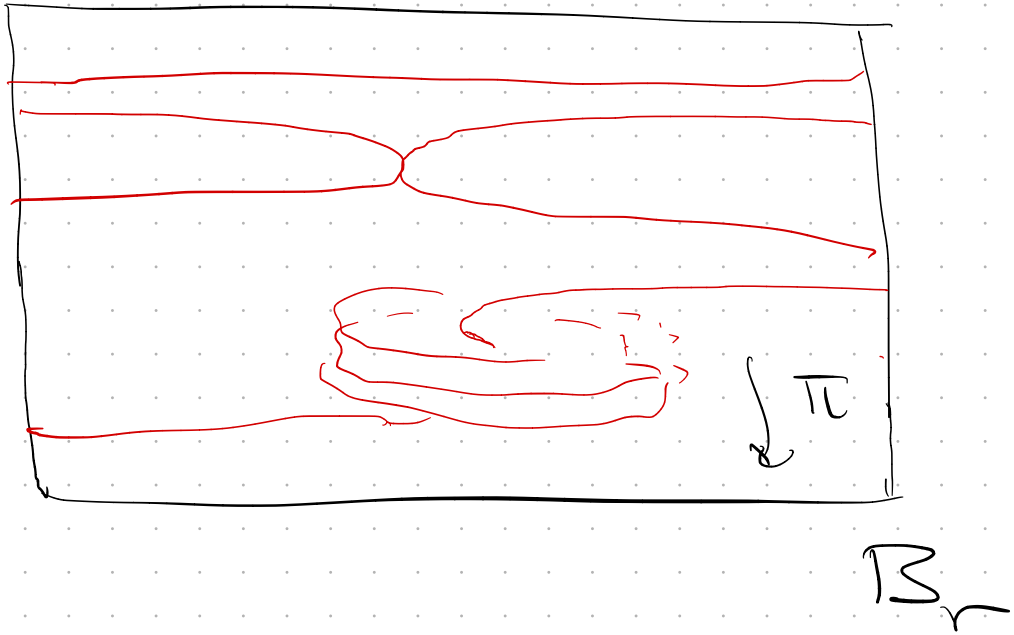
- $(W^u(p) \cap B_r(p))_p$ closed in $B_r(p)$
- Area $\leq B$



Locally proper, bounded area $\Longleftrightarrow$

bounded local folding (geometric) property

"local folding" = degree of π = number of sheets $\leq B$



Lamination $\Longleftrightarrow$ $B = 1$

Thm (Lyubich-Peters) If $\psi \in \hat{\Psi}^u$, then $\psi(\mathbb{C})$ is a nonsingular manifold.

Thm H quasi-expanding, $\psi \in \hat{\Psi}^u$ and $p = \psi(0)$

$$\Longrightarrow \quad \underset{\text{“smooth manifold}}{\psi(\mathbb{C})} \subset \underset{\text{“stable set}}{W^u(p)}.$$

$$\boxed{\psi(\mathbb{C}) = W^u(p) \;?? \qquad \text{"true" unstable manifold}}$$

H is quasi-contracting if H^{-1} is q-expanding

H is quasi-hyperbolic if it is both quasi expanding and quasi contracting.

Thm (B-Guerini-Smillie)
Suppose H is quasi-hyperbolic.
Then

H hyperbolic $\Longleftrightarrow$

No tangency between W^s and W^u

Tangency is the obstruction to hyperbolicity

Thm (B-Guerini-Smillie) H quasihyperbolic
Then:

J^+ laminated in a nbhd of J^*

$\Updownarrow$

J^- laminated in a nbhd of J^*

$\Updownarrow$

H hyperbolic

Unstable slice picture — "The Hubbard Snapshot"

p saddle point

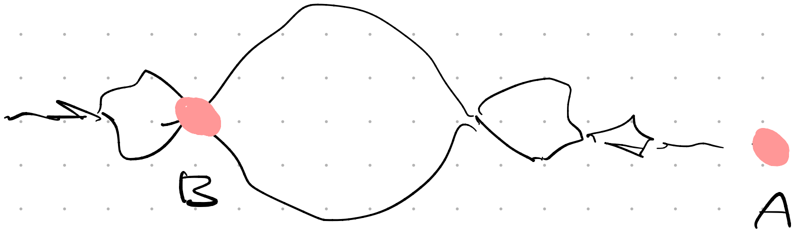
$p \in D \subset W^u(p)$ unstable disk

Computer: draw level sets $\cap$ $G^t|_D$.

unstable slice $D \cap K^t = D \cap \{G^t = 0\}$.

$H(x,y) = (x^2 - 1.1 - .15y,\ x)$

$= $ pertubation of $x \longrightarrow x^2 - 1$ (Basilica)



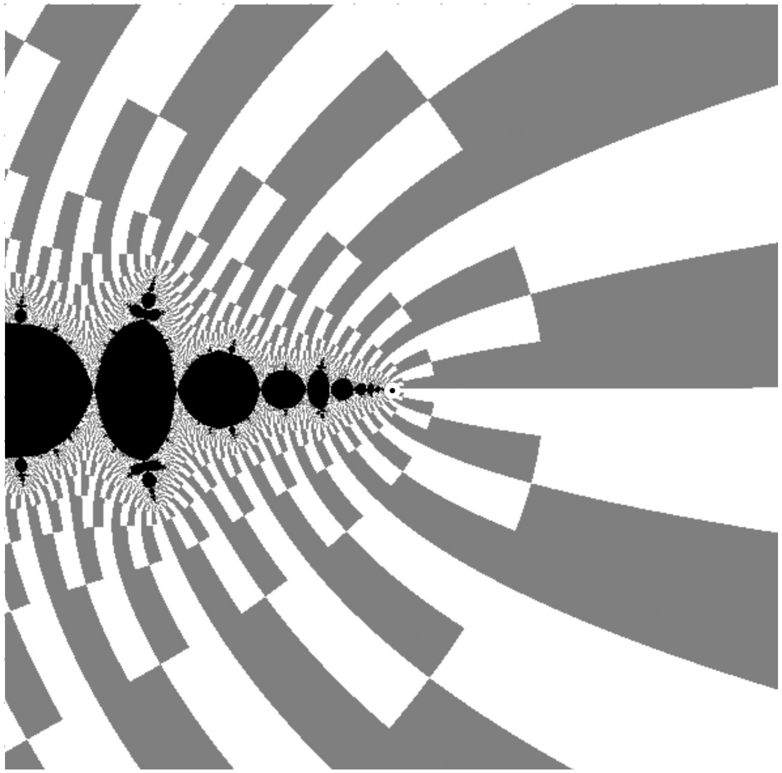
H has saddles

$P_A \sim (A,A)$

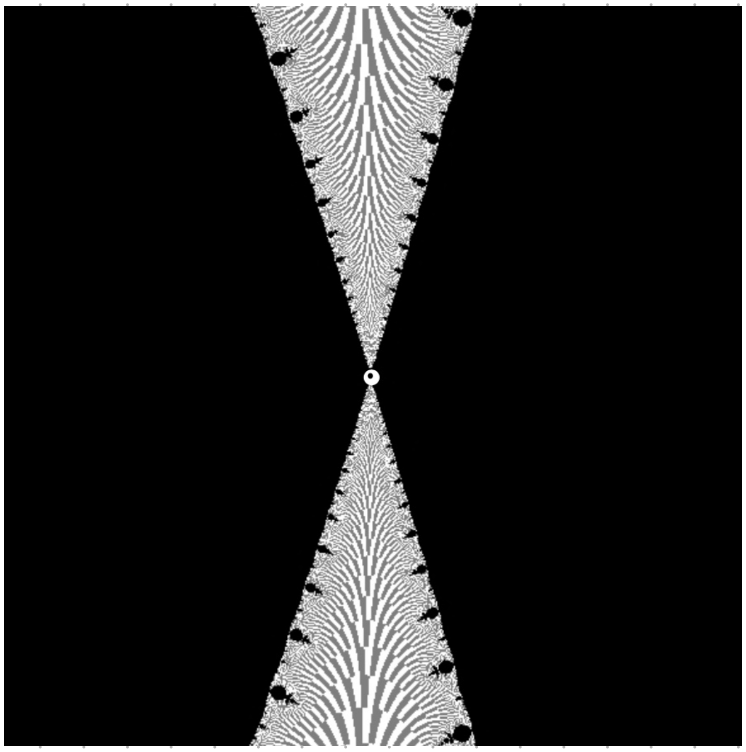
$P_B \sim (B,B)$

unstable slice pictures

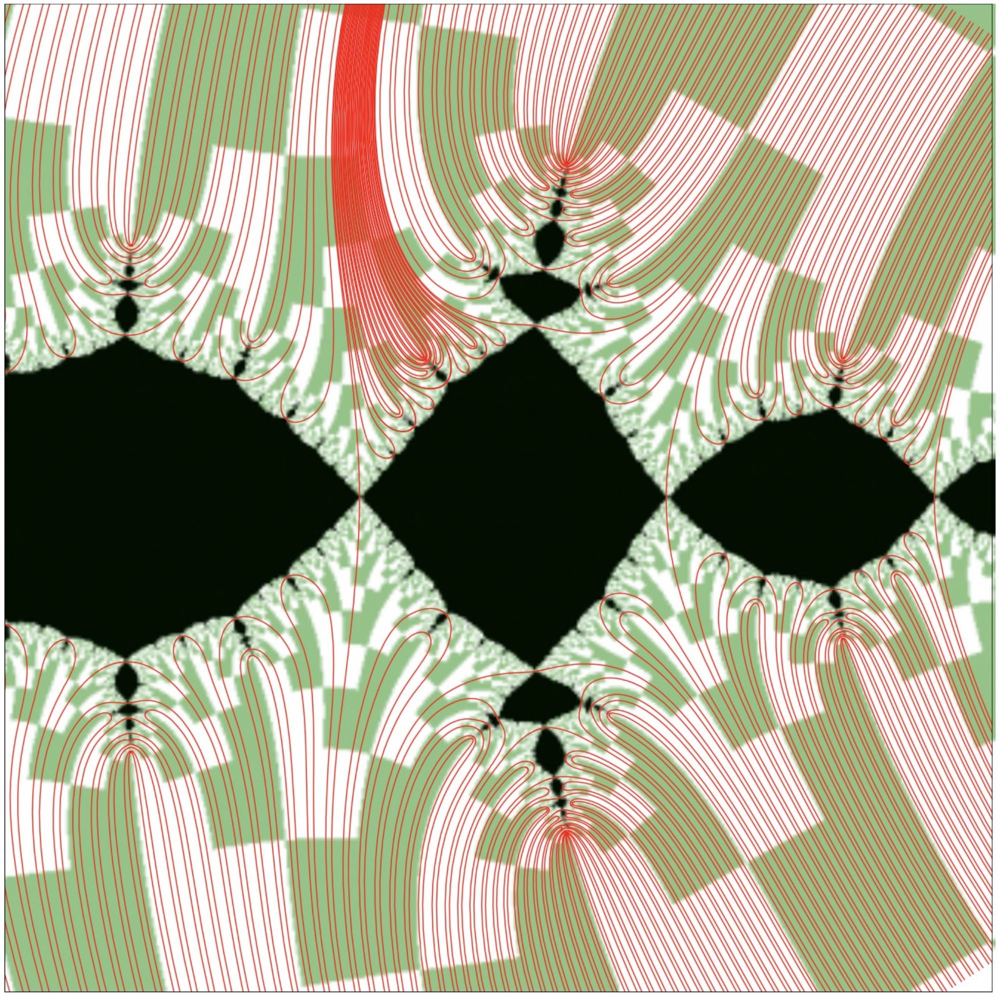
P_A

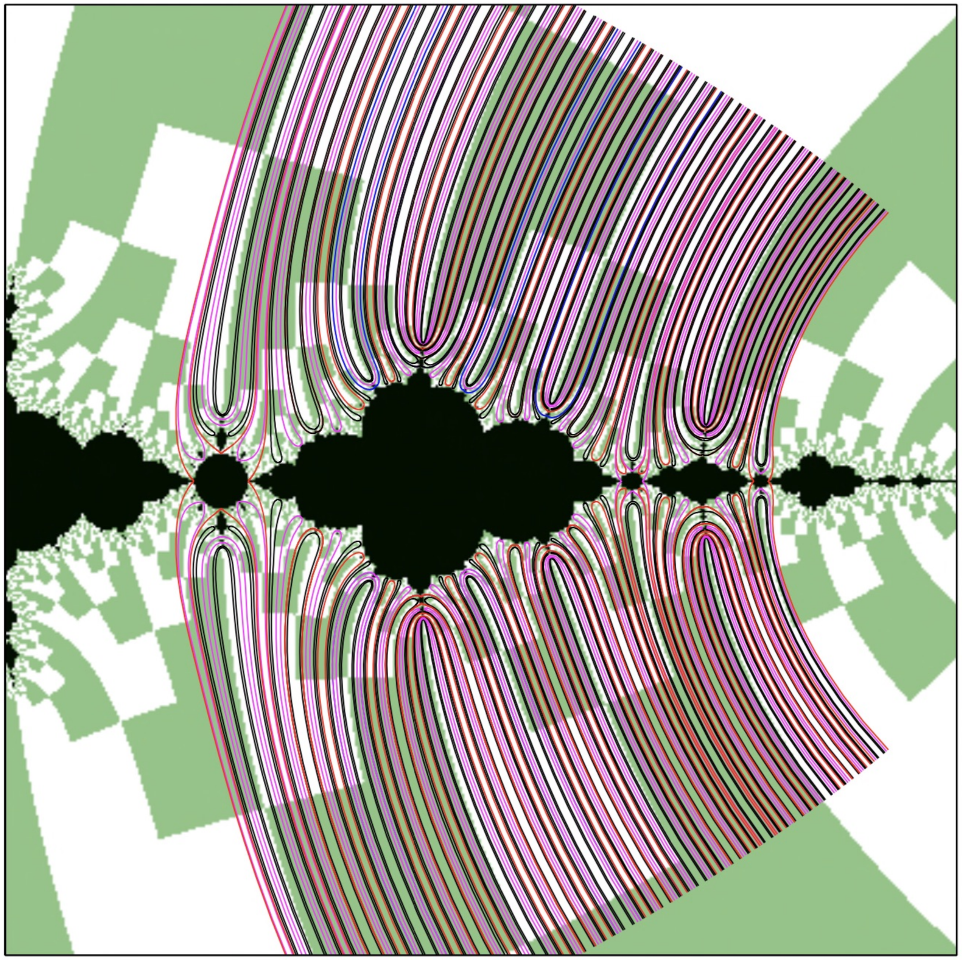


P_B



Gradient lines landing at J (from thesis of R. Oliva, 1998). We can read off the binary solenoidal address of the landing point.





Thm (B-Smillie VI) Suppose H is volume-decreasing, i.e. $|a| < 1$. the following are equivalent:

1. $\exists$ saddle p: the unstable slice $W^u(p)$ is connected
2. J is connected
3. $G^+\big|_{W^u(p) \cap U^+}$ has no critical point.

Thm (Dujardin) J connected $\Longleftrightarrow$ K connect.

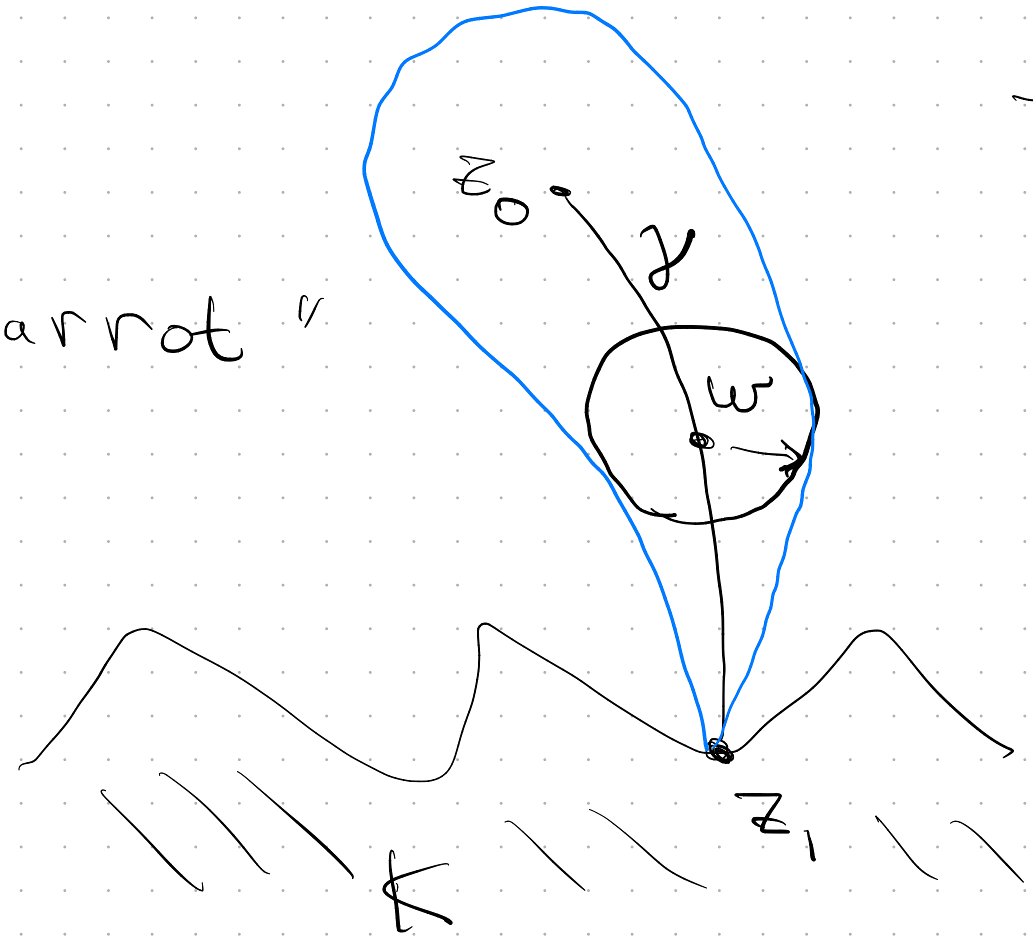
Thus we can "observe" connectivity.

John condition

z_0 = John "center"

$\exists\, \varepsilon > 0 \;\; \forall\, z_1 \in \partial K$
$\exists\, \gamma$ the disk of radius $\varepsilon |w - z_1|$ and center w is disjoint from K.

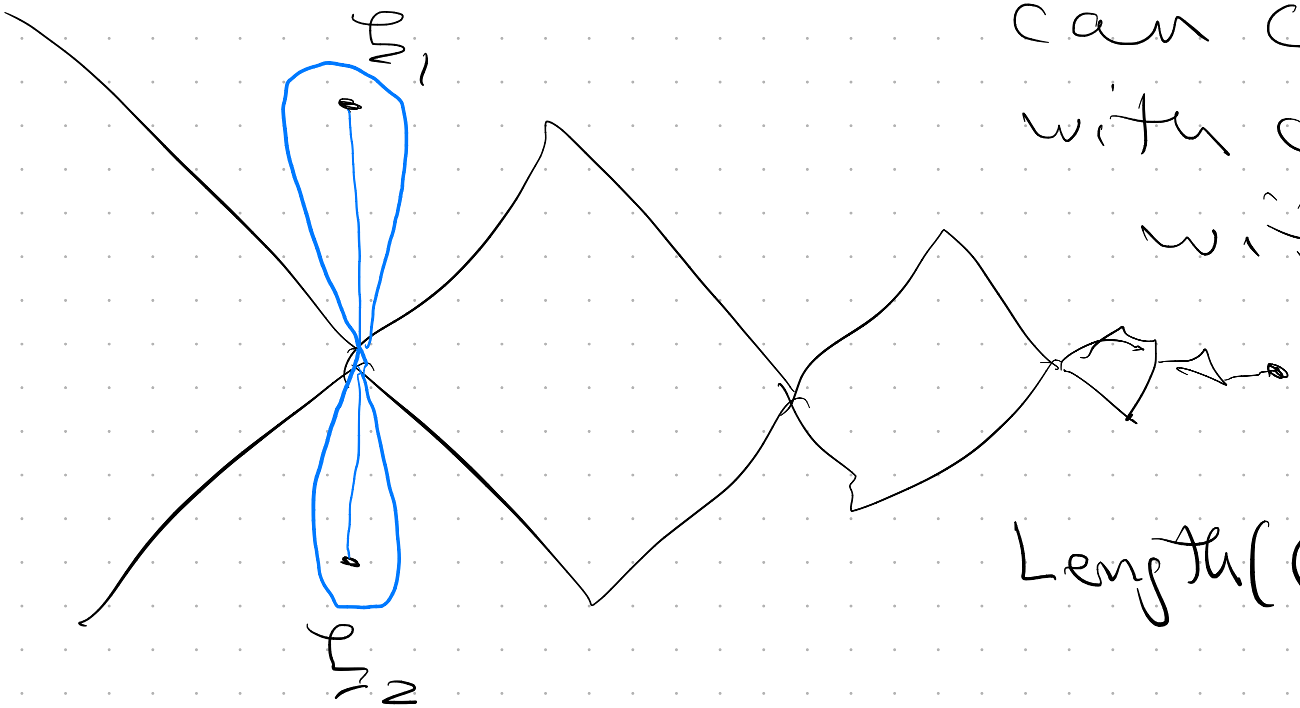
"carrot"



∂K is "open" at each point.

Thm (B-Smillie VII). Suppose H is hyperbolic, and J is connected. Then the unstable slice satisfies the bow-tie condition (variant on John condition)

∄ John center



can connect ξ_1, ξ_2 with carrots C_1, C_2 with

$$\mathrm{Length}(C_j) \leq \mathrm{const}\,|\xi_1 - \xi_2|$$

$\therefore$ The Hubbard snapshot can show the failure of Hyperbolicity

Do quasi-hyperbolic maps satisfy the bowtie condition? ?

Bowtie condition $\Longrightarrow$ quasi-hyperbolicity. ??

Can we describe the shape of $J^{\pm}$, J ?
Is there a model ?

Complex solenoid $\Sigma_+ = \{ \underline{\zeta} = (\zeta_n)_{n\in\mathbb{Z}} :$
$\zeta_n \in \mathbb{C},\ |\zeta_n| > 1,$
$\zeta_{n+1} = \zeta_n^{d} \}$

Real solenoid
$\Sigma_0 = \{ \underline{\zeta} \ \ldots\ldots,\ |\zeta_n| = 1 \ \ldots\ldots \}$

Dissipative (volume decreasing), hyperbolic and J connected :

$J^- \longleftrightarrow \Sigma_+$

$J \longleftrightarrow \Sigma_0/\sim$ finite quotient

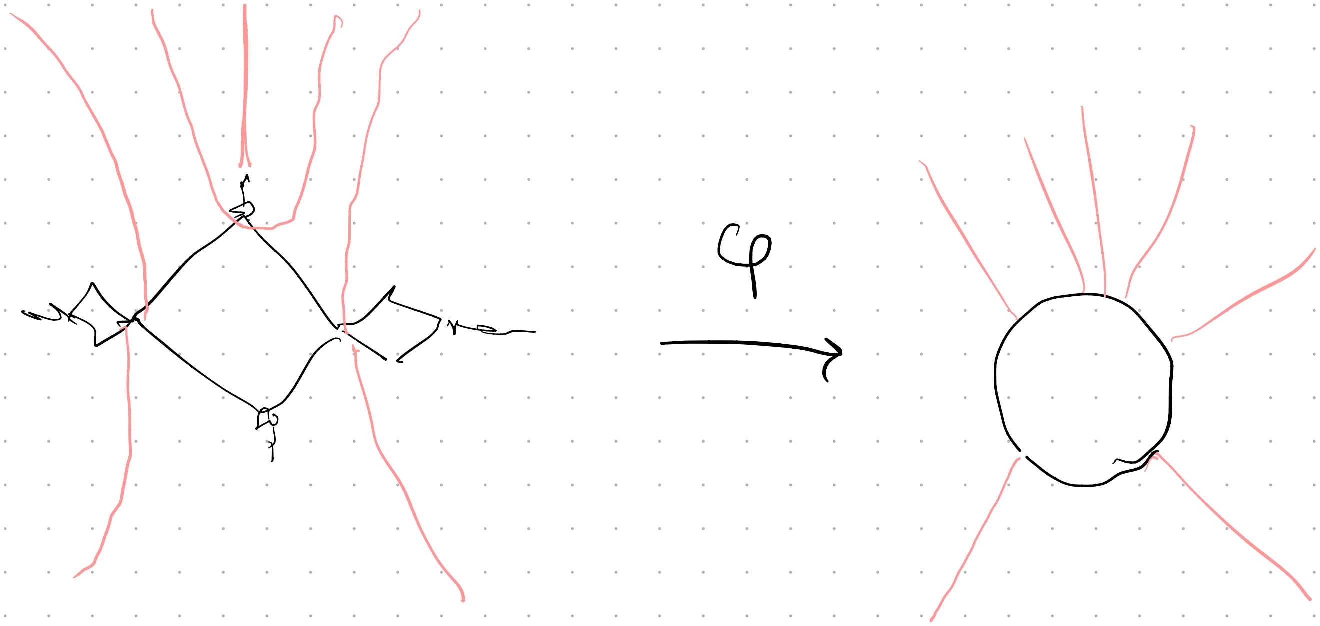
1-D   K connected and hyperbolic

have Böttcher coordinate

$$\varphi : \begin{tikzcd} \mathbb{C}-K \arrow[r, "\cong"] \arrow[loop below, "p"] & \mathbb{C}-\bar{\Delta} \arrow[loop below, "\sigma"] \end{tikzcd} \qquad \sigma(w) = w^d$$

gradient lines of G_K ⟷ radial lines

(external rays)



p hyperbolic $\Longrightarrow$ rays land at J

$J \simeq S^1/\sim$

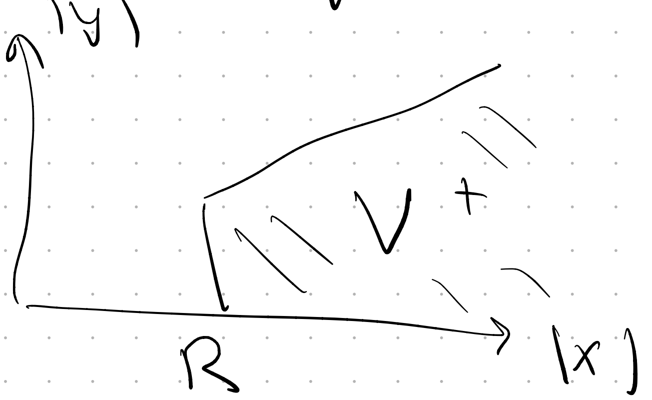
G^+ is pluri-harmonic on $\mathcal{U}^+$, so it has a (local) harmonic conjugate $G^+ + ih$,

Böttcher coordinate $\varphi^+ = e^{G^+ + ih}$

$$\varphi^+ \circ H = (\varphi^+)^d.$$

<u>Thm</u> (Hubbard Oberste-Vorth) may define φ^+ on V^+:

$$\varphi^+ = \lim_{n \to \infty} x_n^{\frac{1}{d^n}}$$



where

$$(x_n, y_n) = H^n(x, y)$$

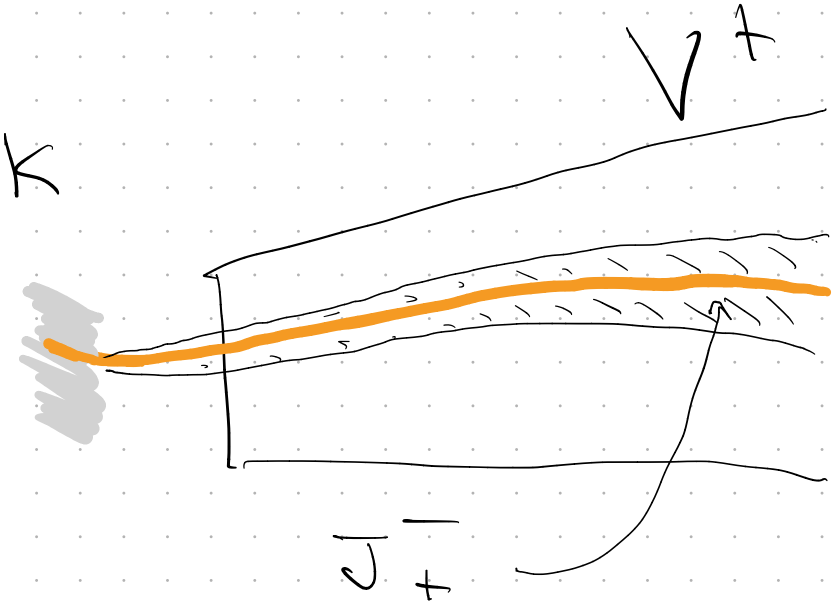
and φ^+ is the root such that

$$\varphi^+ \sim x$$

Thm (B-Smillie VI) If J is connected, then φ^+ has a holomorphic extension to a neighborhood of

$$J_+^- = J^- - K$$

$$= J^- - K^-$$



Have induced map to the complex solenoid

$$\Phi^+ : J_+^- \longrightarrow \Sigma_+$$

$$\Phi^+(p) = \left(\varphi^+(H^n(p))\right)_{n\in\mathbb{Z}}$$

__Thm__ (B-Smillie VII) Suppose H is hyperbolic, J connected. There exists m, $(m,d)=1$ $\sigma_m(\xi) = \xi^m$ and a bijection $\tilde{\Phi} : J_+^- \longrightarrow \Sigma_+$ such that

$$\begin{tikzcd}
J_+^- \arrow[r, shift left, "\tilde{\Phi}"] \arrow[dr, "\Phi^+"'] & \Sigma_+ \arrow[l, shift left, "\Psi"] \arrow[d, "\sigma_m"] \\
 & \Sigma_+
\end{tikzcd}$$

Induced map Ψ takes radial lines in Σ_+ to gradient lines (external rays) inside W^u

$$\Sigma_+ = (1,\infty) \times \Sigma_0$$

The set of rays in Σ_+ is Σ_0

$\{\Psi(r, \underline{s}) ; 1 < r < \infty\}$ is a gradient line for $G^+\big|_{W^u(q)}$

Thm (B-Smillie VII) H hyperbolic, J connected
Gradient line lands at endpoint $e(\underline{s}) \in J$
$e : \Sigma_0 \longrightarrow J$ is continuous, surjective, finite-to-one.

The quotient $\Sigma_0/\sim \,\cong J$ is given by external ray pairs landing at the same point. Look at Oliva's pictures again.

Oliva conjectures the identification pairs for two maps. Computer picture confirms it.

Can you prove it?