On zeros of real random polynomials spanned by OPUC

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Random Polynomials and their Applications

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Kac Polynomials

Theorem (Kac, 1943)

Let η_i be i.i.d. standard real Gaussian random variables and

$$P_n(z) = \eta_0 + \eta_1 z + \cdots + \eta_{n-1} z^{n-1}.$$

Let $N_n(\Omega)$ be the number of zeros of $P_n(x)$ in a measurable set $\Omega \subset \mathbb{R}$. Then,

$$\mathbb{E}(N_n(\Omega)) = \int_\Omega \rho_n^{(1,0)}(x)dx,$$

where

$$\rho_n^{(1,0)}(x) = \frac{1}{\pi}\frac{\sqrt{1-h_n^2(x)}}{|1-x^2|}, \quad h_n(x) = \frac{nx^{n-1}(1-x^2)}{1-x^{2n}}.$$

Moreover, it holds that

$$\mathbb{E}(N_n(\mathbb{R})) = \frac{2+o(1)}{\pi}\log n.$$

Later, in 1995, Shepp and Vanderbei have derived a formula for $\rho_n^{(0,1)}(z)$, the intensity function for the complex zeros of $P_n(z)$.

Random Linear Combinations

Theorem (Edelman and Kostlan, 1995)

Let $f_i(z)$ be arbitrary function in $\mathbb{C}$ that are real and differentiable on $\mathbb{R}$ and

$$P_n(z) = \eta_0 f_0 + \eta_1 f_1(z) + \cdots + \eta_{n-1} f_{n-1}(z),$$

where η_i are elements of a multivariate real normal distribution with mean 0 and covaraince matrix C, then

$$\rho_n^{(1,0)}(x) = \frac{1}{\pi} \frac{\partial^2}{\partial t \partial s} \log \left(v(s)^\mathsf{T} C v(t) \right) \Big|_{s=t=x},$$

where $v(x) = (f_0(x), \ldots, f_{n-1}(x))^\mathsf{T}$. If η_i are i.i.d. real Gaussians, then

$$\rho_n^{(1,0)}(x) = \frac{1}{\pi} \frac{\sqrt{K_n(x,x) K_n^{(1,1)}(x,x) - K_n^{(1,0)}(x,x)^2}}{K_n(x,x)}, \qquad (*)$$

where $K_n^{(i,j)}(z,w) = \sum_{k=0}^{n-1} f_k^{(i)}(z) \overline{f_k^{(j)}(w)}$.

Independent proofs of $(*)$ by Lubinsky, Pritsker, Xie (2016) and Vanderbei (2016).

Proof by Vanderbei

Let Ω be a domain with rectifiable boundary. By the Cauchy's formula

$$\mathbb{E}(N_n(\Omega)) = \int_{\partial\Omega} \mathbb{E}\frac{P_n'(z)}{P_n(z)}\,dz.$$

Let C be the covariance matrix of $\xi = (\Re P_n(z), \Im P_n(z), \Re P_n'(z), \Im P_n'(z))^\top$. Then,

$$\xi \overset{D}{=} L\tau, \quad C = LL^\top,$$

where L is lower triangular and $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)^\top$ with τ_i i.i.d. Gaussians. Then,

$$\mathbb{E}\frac{P_n'(z)}{P_n(z)} = \mathbb{E}\frac{\alpha\tau_1 + \beta\tau_2}{\gamma\tau_1 + \delta\tau_2} = \frac{\alpha}{\delta} f\left(\frac{\gamma}{\delta}\right) + \frac{\beta}{\gamma} f\left(\frac{\delta}{\gamma}\right), \quad f(w) = \mathbb{E}\frac{\tau_1}{w\tau_1 + \tau_2}.$$

Since $\tau_1 + \mathrm{i}\tau_2$ is a complex Gaussian random variable, one can easily compute

$$\frac{1}{f(w)} = \begin{cases} w + \mathrm{i}, & \Im w > 0, \\ w - \mathrm{i}, & \Im w < 0. \end{cases}$$

Collapsing Ω to a subset of $\mathbb{R}$, leads to $(*)$, where $K_n^{(i,j)}(z,w)$ appear in L.

Theorem (Newland and Ya.)

Let $f_i(z)$ be arbitrary entire function that are real on $\mathbb{R}$ and

$$P_n(z) = \eta_0 f_0 + \eta_1 f_1(z) + \cdots + \eta_{n-1} f_{n-1}(z),$$

where η_i be i.i.d. standard real Gaussian random variables. Let $N_n^y(\Omega)$ be the number of solutions of $P_n(x) = y$ in a measurable set $\Omega \subset \mathbb{R}$. Then,

$$\mathbb{E}\left(N_n^{\sqrt{2}y}(\Omega)\right) = \frac{1}{\pi}\int_\Omega \frac{E_n(x)}{K_n(x)} \exp\left(-\frac{K_n^{(1,1)}(x)}{E_n^2(x)} y^2\right) dx +$$

$$\frac{2|y|}{\pi}\int_\Omega \frac{|K_n^{(1,0)}(x)|}{K_n^{3/2}(x)} \exp\left(-\frac{y^2}{K_n(x)}\right) \operatorname{erf}\left(|y|\frac{|K_n^{(1,0)}(x)|}{K_n(x)E_n(x)}\right) dx,$$

where $K_n^{(i,j)}(x) = K_n^{(i,j)}(x,x)$ and $E_n^2(x) = K_n(x)K_n^{(1,1)}(x) - K_n^{(1,0)}(x)^2$.

The proof follows Vanderbei's argument, expression for $f_y(w)$ is more complicated.

It is a generalization of Farahmand (monomials 1986 and some other families).

Orthogonal Polynomials on the Unit Circle

Let μ be a Borel measure on $\mathbb{T}$ symmetric w.r.t. conjugation and $\varphi_i(z)$ be the corresponding orthonormal polynomials

$$\int_{\mathbb{T}} \varphi_i(z)\overline{\varphi_j(z)}d\mu(z) = \delta_{ij}.$$

In this case $\varphi_i(z)$ has real coefficients. Write $\varphi_i(z) = \kappa_i \Phi_i(z)$, where $\Phi_i(z)$ is monic.

$$\Phi_{i+1}(z) = z\Phi_i(z) - \alpha_i \Phi_i^*(z),$$
$$\Phi_{i+1}^*(z) = \Phi_i(z) - \alpha_i z \Phi_i(z),$$

where $\alpha_i \in (-1, 1)$ and $\Phi_i^*(z) = z^i \Phi_i(1/z)$. It holds that

$$K_n(z, w) = \frac{\varphi_n^*(z)\varphi_n^*(\bar{w}) - \varphi_n(z)\varphi_n^*(\bar{w})}{1 - z\bar{w}}.$$

When μ is the normalized Lebesgue measure on $\mathbb{T}$, $\varphi_i(z) = z^j$.

Intensity Functions

Theorem (Ya. and Yeager, 2019)

Let $f_i(z) = \varphi_i(z)$. Then,

$$\rho_n^{(1,0)}(x) = \frac{1}{\pi} \frac{\sqrt{1 - h_n^2(x)}}{|1 - x^2|}, \quad h_n(x) = \frac{(1 - x^2) b_n'(x)}{1 - b_n^2(x)},$$

where $b_n(z) = \varphi_n(z)/\varphi_n^*(z)$. If $\alpha_n \to 0$ as $n \to \infty$ (Nevai class), then

$$\rho_n^{(1,0)}(x) \to \frac{1}{\pi} \frac{1}{|1 - x^2|}$$

locally uniformly $\mathbb{R} \backslash \{\pm 1\}$. Moreover, in this case

$$\rho_n^{(0,1)}(z) \to \frac{1}{\pi (1 - |z|^2)^2} \sqrt{1 - \left| \frac{1 - |z|^2}{1 - z^2} \right|^2}$$

locally uniformly in $\overline{\mathbb{C}} \backslash (\mathbb{T} \cup \mathbb{R})$.

Expected Number of Real Zeros

μ is called doubling on a subarc $T \subset \mathbb{T}$ if $\mu(2I) \leqslant c\mu(I)$ for any $2I \subset T$.

Theorem (Ya. and Yeager, 2019)

Let η_i be i.i.d. standard real Gaussian random variables and

$$P_n(z) = \eta_0\varphi_0(z) + \eta_1\varphi_1(z) + \cdots + \eta_{n-1}\varphi_{n-1}(z).$$

If there exist two subarcs of $\mathbb{T}$ centered at ± 1 on which μ is doubling, then

$$\mathbb{E}(N_n(\mathbb{R})) \leqslant \frac{2}{\pi}\log n + O(1).$$

If numbers $n^p|\alpha_n|$ are uniformly bounded for some $p > \frac{3}{2}$, then

$$\mathbb{E}(N_n(\mathbb{R})) = \frac{2 + o(1)}{\pi}\log n.$$

The second condition implies that μ absolutely continuous w.r.t. $|dz|$ and the Radon-Nikodym derivative is continuous and non-vanishing.

Expected Number of Real Zeros

Proposition (Ya. and Yeager, 2019)

Let $\mu = t\nu + (1-t)\delta_1$, where ν is a conjugate-symmetric Borel measure such that the numbers $n^p|\alpha_n(\nu)|$ are uniformly bounded for some $p > \frac{3}{2}$. Then, $n\alpha_n(\mu) \sim 1$ and

$$\mathbb{E}(N_n(\mathbb{R})) = \frac{2+o(1)}{\pi}\log n.$$

It is unclear what is the weakest condition on μ that allows to keep Kac asymptotics.

Upper Bound

It holds that

$$\mathbb{E}(N_n(\mathbb{R})) \leqslant \frac{2}{\pi}\log n + \mathcal{O}(1) + \frac{2}{\pi}\left(\int_{-1}^{-1+1/n} + \int_{1-1/n}^{1}\right)\sqrt{\frac{K_n^{(1,1)}(x,x)}{K_n(x,x)}}dx$$

and

$$\int_{1-1/n}^{1}\sqrt{\frac{K_n^{(1,1)}(x,x)}{K_n(x,x)}}dx = \int_0^1\sqrt{\frac{K_n^{(1,1)}(1-y/n,1-y/n)}{n^2K_n(1-y/n,1-y/n)}}dy.$$

Upper Bound

Proposition (Ya. and Yeager, 2019)

If μ is doubling on T, $T' \subset T$ is a proper subarc, and $|a| \leqslant 2$, then

$$K_n\left(ze^{\mathrm{i}a/n}, ze^{\mathrm{i}a/n}\right) \sim \mu_n(z) := \int_{T(z,1/n)} d\mu$$

uniformly w.r.t. z, a, n (if μ is doubling on $\mathbb{T}$, Mastroianni and Totik, 2000).

Thus,

$$K_n\left(1+\frac{u}{n}, 1+\frac{u}{n}\right) \sim \frac{1}{\mu_n(1)},$$

by Cauchy-Schwarz

$$\left|K_n\left(1+\frac{u}{n}, 1+\frac{\bar{v}}{n}\right)\right| \lesssim \frac{1}{\mu_n(1)},$$

and by Cauchy integral formula

$$\left|K_n^{(1,1)}\left(1+\frac{u}{n}, 1+\frac{\bar{v}}{n}\right)\right| \lesssim \frac{n^2}{\mu_n(1)}.$$

Lower Bound

It holds that

$$\mathbb{E}(N_n(\mathbb{R})) \geqslant \frac{2}{\pi}\int_{-1+\log n/n}^{1+\log n/n} \frac{\sqrt{1-h_n^2(x)}}{1-x^2}dx \leqslant \frac{2}{\pi}\log\frac{n}{\log n}\sqrt{1-M_n^2},$$

where M_n is the maximum of $|h_n(x)|$ on the integration interval, where

$$h_n(x) = \frac{(1-x^2)b_n'(x)}{1-b_n^2(x)}.$$

Since $\Phi_n^*(z) = 1 - z\sum_{k=0}^{n-1}\alpha_k\Phi_k(z)$, it is simple to show that

$$|b_n(z)| \lesssim |z|^{n-m} + \sum_{k=m}^{\infty}|\alpha_k|, \quad |z| \leqslant 1,$$

for any $m \leqslant n-1$. From this one can deduce that

$$M_n \lesssim (\log n)^{p-1}n^{3/2-p}.$$

Theorem (Ya. and Yeager, 2019)

Assume that μ is Ullman-Stahl-Totik regular, that is,

$$\varepsilon_n^2 := \frac{1}{n} \log \kappa_n = -\frac{1}{2n} \sum_{i=0}^{n-1} \log\left(1 - \alpha_i^2\right) \to 0$$

as $n \to \infty$. Given subarc $T \subset \mathbb{T}$ ad $\delta \in (0,1)$, it holds that

$$\mathbb{E}\left(\left|\frac{1}{n} N_n(\Omega(T,\delta)) - \frac{|T|}{2\pi}\right|\right) \lesssim \frac{1}{\delta}\sqrt{\frac{\log n}{n}} + \varepsilon_n,$$

where $\Omega(T,\delta) = \{rz : z \in T,\ r \in (1-\delta, 1+\delta)\}$.

The proof is an improvement of Pritsker and Yeager 2015, which had a more stringent assumption on μ.

Theorem (Ya. and Yeager, 2019)

Assume that $\alpha_n \to 0$ as $n \to \infty$. Let $S \subset \mathbb{T}\backslash\{\pm 1\}$ be compact while μ is absolutely continuous on an open set containing S and the Radon-Nikodym derivative is positive and continuous on S. Then,

$$\frac{1}{n}\mathbb{E}\left(N_n(\Omega(S,\tau_1,\tau_2))\right) \to \frac{|S|}{2\pi}\left(\frac{H'(\tau_2)}{H(\tau_2)} - \frac{H'(\tau_1)}{H(\tau_1)}\right)$$

where $\Omega(S,\tau_1,\tau_2) = \{rz : z \in S,\ r \in (1+\frac{\tau_1}{2n}, 1+\frac{\tau_2}{2n})\}$ and $H(\tau) = \frac{e^\tau - 1}{\tau}$.

It holds that

$$\lim_{\tau \to -\infty} \frac{H'(\tau)}{H(\tau)} = 0 \quad \text{and} \quad \frac{H'(\tau)}{H(\tau)} = 1 - \frac{H'(-\tau)}{H(-\tau)}.$$

This theorem is a consequence of the universality results of Levin and Lubinsky 2007 concerning reproducing kernels of OPUC.

Theorem (Wilkins, 1999)

In the case of Kac polynomials it holds that

$$\mathbb{E}(N_n(\mathbb{R})) \sim \frac{2}{\pi}\log n + A_0 + \sum_{p=1}^{\infty}\frac{A_p}{n^p},$$

where $f^2(t) := 1 - t^2 \operatorname{csch}^2 t$,

$$A_0 = \frac{2}{\pi}\left(\log 2 + \int_0^1 t^{-1} f(t)dt + \int_1^{\infty} t^{-1}(f(t)-1)dt\right)$$

and $A_1 = A_3 = A_5 = 0$.

Real Zeros

Theorem (Aljubran and Ya., 2019)

Let μ be absolutely continuous on $\mathbb{T}$ whose Radon-Nikodym derivative extends to a non-vanishing holomorphic function in some neighborhood of $\mathbb{T}$. Then,

$$\mathbb{E}(N_n(\mathbb{R})) \sim \frac{2}{\pi}\log n + A_0 + \sum_{p=1}^{\infty} \frac{A_p^{\mu}}{n^p}.$$

Strong Asymptotics

Let $D_{int}(z)$ and $D_{ext}(z)$ be the restrictions to $\mathbb{D}$ and $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$ of the Szegő function

$$D(z) := \exp\left(\frac{1}{4\pi}\int_{\mathbb{T}}\frac{\xi+z}{\xi-z}\log\mu'(\xi)|d\xi|\right).$$

Then, both $D_{int}(z)$ and $D_{ext}(z)$ extend analytically across $\mathbb{T}$. Moreover,

$$b_n(z) = z^n S(z)\,\frac{\mathcal{E}_n(z) - \tau^2 z^{-n} S^{-1}(z)\mathcal{I}_n(z)}{\mathcal{E}_n(1/z) - \tau^2 z^n S(z)\mathcal{I}_n(1/z)},$$

where $S(z) := D_{int}(z)D_{ext}(z)$ and $\tau := D_{ext}(\infty)$. It also holds that

$$|\mathcal{E}_n(z) - 1| \leqslant \frac{C_s s^{2n}}{1/s - |z|} \quad \text{and} \quad |\mathcal{I}_n(z)| \leqslant \frac{C_s s^n}{|z| - s}$$

for $s < |z| < 1/s$. This allows to extend Wilkins' argument for $b_n(z) = z^n$.

Geronimus Polynomials

Geronimus polynomials $\varphi_i(z;\alpha)$ are OPUC corresponding to $\alpha_n = \alpha \in \mathbb{D}$. Their measure of orthogonality is explicitly known and is supported on

$$\Delta_\alpha := \{e^{i\theta} : 2\arcsin(|\alpha|) \leqslant \theta \leqslant 2\pi - 2\arcsin(|\alpha|)\}$$

with a pure mass point that is present if and only if $|\alpha + \frac{1}{2}| > \frac{1}{2}$. For real α the mass point is at 1 and is present if and only if $\alpha > 0$, in which case $\varphi_i(z;\alpha)$ has a zero exponentially close to 1.

Theorem (Aljubran and Ya., 2021)

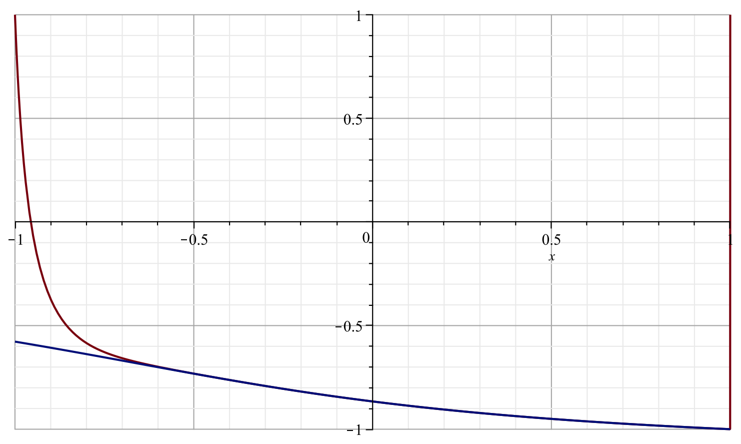
Define $r(z) = \sqrt{(z-1)^2 + 4\alpha^2 z}$ to be the branch holomorphic in $\mathbb{C}\backslash\Delta_\alpha$ such that $r(z)/z \to 1$ as $z \to \infty$. Then,

$$b_n(z) \to \frac{-2\alpha}{r(z) + 1 - z}$$

locally uniformly in $\mathbb{D}$. Moreover,

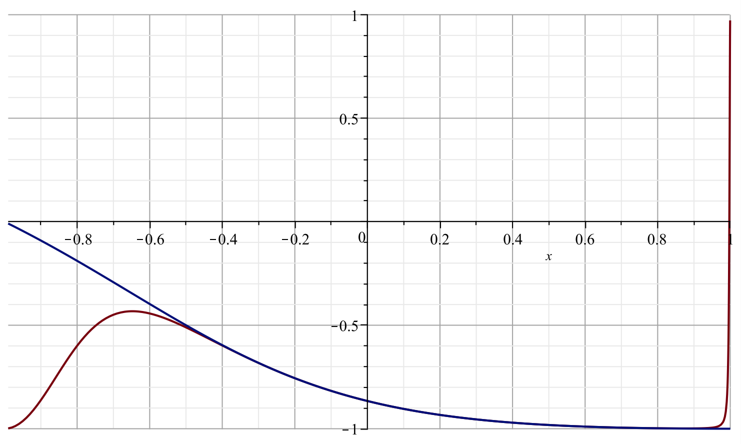
$$h_n(x) = -\alpha \frac{x+1}{r(x)} \left(1 + \mathcal{O}\left((1-x)^2 n e^{-\sqrt{n/(1-\alpha^2)}}\right)\right)$$

for $-1 + 1/\sqrt{n} \leqslant x \leqslant 1 - \delta_\alpha^{n+1}$, where $\delta_\alpha = 0$, $\alpha < 0$, and $\delta_\alpha^3 = \frac{1-\alpha}{1+\alpha}$, $\alpha > 0$.



The graphs of $b_4(x)$ and $\frac{-2\alpha}{r(x)+1-x}$ for $\alpha = \sqrt{3}/2$.

Geronimus Polynomials



The graphs of $h_4(x)$ and $-\alpha \frac{x+1}{r(x)}$ for $\alpha = \sqrt{3}/2$.

Theorem (Aljubran and Ya., 2021)

Fix $\alpha \in (-1,1)\backslash\{0\}$. Let η_i be i.i.d. standard real Gaussian random variables and

$$P_n(z) = \eta_0\varphi_0(z;\alpha) + \eta_1\varphi_1(z;\alpha) + \cdots + \eta_{n-1}\varphi_{n-1}(z;\alpha).$$

Then,

$$\mathbb{E}(N_n(\mathbb{R})) \sim \frac{1}{\pi}\log n + A_0^\alpha + \sum_{p=1}^{\infty} \frac{A_p^{\alpha,(-1)^n}}{n^p},$$

where

$$A_0^\alpha = \frac{A_0 + 1 + \operatorname{sgn}(\alpha)}{2} + \frac{1}{\pi}\log\frac{2}{|\alpha|}.$$

$A_0^{|\alpha|} = A_0^{-|\alpha|} + 1$ due to the zeros of $\varphi_i(z;|\alpha|)$ close to 1 while $\varphi_i(z;-|\alpha|)$ do not have such zeros.