What do Painlevé equations have in common with graph enumeration on Riemann surfaces?

Maxim L. Yattselev

INDIANAPOLIS **SCHOOL OF SCIENCE**
Department of Mathematical Sciences

Colloquium

University of Missouri

October 17th, 2024

Classical Special Functions

Throughout 18-20th centuries special functions appeared in many physical problems:

- Airy function (1838) was introduced during study of intensity of light
- Bessel functions appeared in the work of Jakob Bernoulli on the oscillations of heavy chains (1738), of Euler on the vibration of circular membrane (1764), and of Lagrange on elliptic motion (1769)
- parabolic cylinder functions appear while solving Helmholtz equation (eigenvalues of the Laplacian)
- hypergeometric functions appear while solving one-dimensional quantum scattering through and over barriers
- confluent hypergeometric functions appear while solving wave equation

All of these functions are solutions of second order linear differential equations.

Bessel Function

Bessel functions can be defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\Gamma(\nu + k + 1)}$$

and solve

$$z^2 y'' + zy' + (z^2 - \nu^2)y = 0.$$

It holds that

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z \quad \text{and} \quad J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z.$$

Jacobi Polynomials

Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, themselves hypergeometric functions, can be defined either via orthogonality

$$\int_{-1}^{1} x^k P_n^{(\alpha,\beta)}(x)(1+x)^\beta (1-x)^\alpha dx = 0,$$

$k = \overline{0, n-1}$, $\alpha, \beta > -1$, or via differential equation

$$(1-x^2)y'' + [\beta - \alpha - (\beta+\alpha+2)x]y' + n(n+\alpha+\beta+1)y = 0.$$

$P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$ "=" $\cos(n \arccos x)$ and $P_n^{(\frac{1}{2},\frac{1}{2})}(x)$ "=" $\dfrac{\sin((n+1)\arccos x)}{\sin(\arccos x)}$.

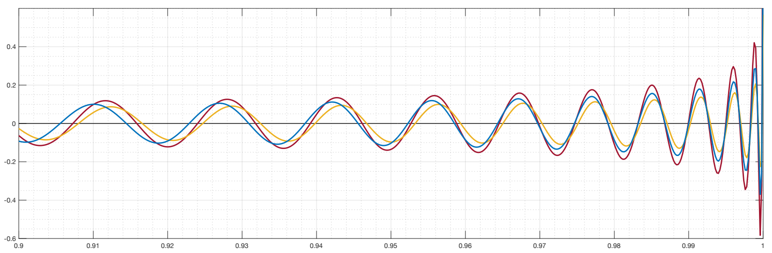
Theorem (Stieltjes)

The zeros of $P_n^{(2q-1,2p-1)}(x)$ are the unique minimizer on $[-1,1]$ of the weighted logarithmic energy

$$\sum_{i=1}^{n} \left(p \log \frac{1}{1-x_i} + q \log \frac{1}{1+x_i} \right) + \sum_{i<j} \log \frac{1}{|x_i - x_j|}$$

Theorem (Mehler-Heine Formula)

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(1-\frac{z^2}{2n^2}\right) = \left(\frac{2}{z}\right)^{\alpha} J_\alpha(z).$$



$P_{150}^{(.1,0)}(x)$ (red), $P_{150}^{(-.1,0)}(x)$ (yellow), and $J_0(z)$ (blue) on $[.9, 1]$.

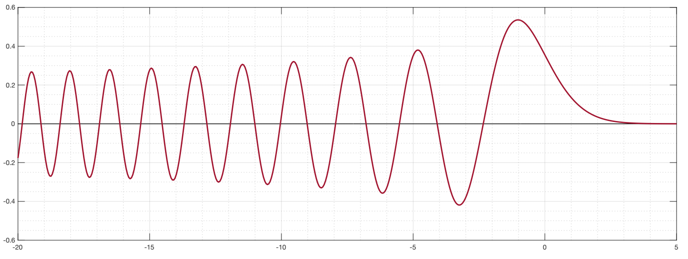
Airy Functions

Airy functions $\mathrm{Ai}(z), \mathrm{Bi}(z)$ solve differential equation

$$y'' - zy = 0$$

and are related to Bessel functions via $\zeta = (2/3)z^{3/2}$,

$$\mathrm{Ai}(-z) = \sqrt{\frac{z}{3}}\left(J_{-1/3}(\zeta) + J_{1/3}(\zeta)\right), \quad \mathrm{Bi}(-z) = \sqrt{\frac{z}{3}}\left(J_{-1/3}(\zeta) - J_{1/3}(\zeta)\right).$$



Airy Functions

Functions $\mathrm{Ai}(z), \mathrm{Bi}(z)$ admit integral representations

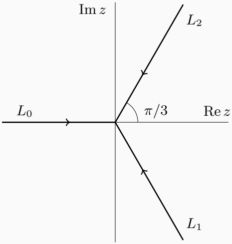
$$\mathrm{Ai}(z) = \frac{1}{2\pi\mathrm{i}} \int_{L_1 - L_2} e^{-V(s;z)} ds$$

and

$$\mathrm{Bi}(z) = \frac{1}{2\pi} \left(\int_{L_0 - L_1} + \int_{L_0 - L_2} \right) e^{-V(s;z)} ds,$$

where $L_k = \{x e^{\pi\mathrm{i}(-1+2k/3)} : x \in (0, \infty)\}$ oriented towards the origin and

$$V(s;z) = -\frac{s^3}{3} + zs.$$



Hermite Polynomials

Hermite polynomials $H_n(z)$ can be defined either via orthogonality

$$\int_{-\infty}^{\infty} x^k H_n(x) e^{-x^2} dx = 0,$$

$k = \overline{0, n-1}$ or via differential equation

$$y'' - 2zy' + 2ny = 0.$$

Theorem (Plancherel-Rotach)

For $x = \sqrt{2n+1} - t/a_n$, it holds that

$$e^{-x^2/2} H_n(x) = c_n \left(\mathrm{Ai}(t) + \mathcal{O}(n^{-2/3}) \right),$$

where $a_n = 2^{1/2}\, 3^{1/3}\, n^{1/6}$ and $c_n = \pi^{-3/4} 3^{1/3} 2^{n/2+1/4} (n!)^{1/2} n^{-1/12}$.

Painlevé Equations

At the beginning of the 20th century, it was shown by Painlevé and Gambier that among the second order differential equations of the form

$$y'' = F(z, y, y'),$$

where F is rational in y, y' and analytic in z, there are exactly 50 canonical equations whose solutions do not possess movable branch points. Among these, 44 can either be reduced to linear equations, equations solved in terms of elliptic functions, or to the remaining 6 equations.

Painlevé Equations

$$\mathrm{P_I} \quad y'' = 6y^2 + z,$$

$$\mathrm{P_{II}} \quad y'' = 2y^3 + zy + \alpha,$$

$$\mathrm{P_{III}} \quad y'' = \frac{1}{y}(y')^2 - \frac{y'}{z} + \frac{\alpha y^2 + \beta}{z} + \gamma y^3 + \frac{\delta}{y},$$

$$\mathrm{P_{IV}} \quad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \alpha)y + \frac{\beta}{y},$$

$$\mathrm{P_V} \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{z} + \frac{(y-1)^2}{z^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{z} + \delta\frac{y(y+1)}{y-1},$$

$$\mathrm{P_{VI}} \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)(y')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y' + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left(\alpha + \beta\frac{z}{y^2} + \gamma\frac{z-1}{(y-1)^2} + \delta\frac{z(z-1)}{(y-z)^2}\right).$$

Longest Increasing Subsequence

Let $\pi \in S_N$ be a permutation of $1, 2, \ldots, N$. $\pi(m_1), \pi(m_2), \ldots, \pi(m_n)$ is said to be an increasing subsequence if

$$1 \leq m_1 < m_2 < \cdots < m_n \leq N \quad \text{and} \quad \pi(m_1) < \pi(m_2) < \cdots < \pi(m_n).$$

Theorem (Baik-Deift-Johansson)

Equip S_N with uniform distribution. Denote by $\ell_N(\pi)$ the length of the longest increasing subsequence of $\pi \in S_N$. Then

$$\lim_{N \to \infty} \text{Prob}\left(\frac{\ell_N(\pi) - 2\sqrt{N}}{N^{1/6}} \leq s\right) = F_2(s),$$

where $F_2(s)$ is the Tracy-Widom distribution given by

$$F_2(s) = \exp\left\{-\int_s^\infty (x - s) y^2(x) dx\right\}$$

and $y(x)$ is the Hastings-McLeod solution of P_{II}: $\alpha = 0$ and

$$y(x) \sim \mathrm{Ai}(x), \quad x \to \infty, \qquad y(x) \sim \sqrt{-x/2}, \quad x \to -\infty.$$

Special Solutions

For certain values of the parameters, equations $P_{II} - P_{VI}$ have rational and algebraic solutions as well as special solutions which are expressed via classical special functions.

$P_{II} - P_{VI}$ also have Bäcklund transformations that map solutions of a given Painlevé equation to solutions of the same Painlevé equation, but with different values of the parameters.

Airy Solutions of $\mathrm{P_{II}}$

When $\alpha = 1/2$, $\mathrm{P_{II}}$ has a solution

$$q_1(z;\lambda) = -\frac{d}{dz}\log\left(C_1\mathrm{Ai}\left(-2^{-1/3}z\right) + C_2\mathrm{Bi}\left(-2^{-1/3}z\right)\right),$$

where $\lambda = C_2/C_1 \in \overline{\mathbb{C}}$. Using Bäcklund transformations one then constructs

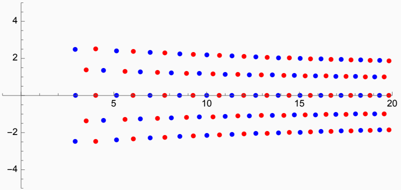
$$q_{n+1}(z;\lambda) = -q_n(z;\lambda) - \frac{2n}{2q_n^2(z;\lambda) + 2q_n'(z;\lambda) + z}$$

solutions of $\mathrm{P_{II}}$ for $\alpha = (n+1) - 1/2$, $n \geq 1$, as well as

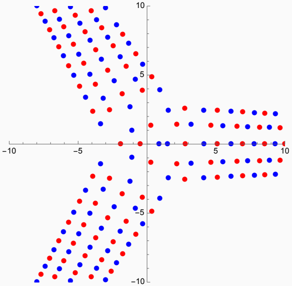
$$q_{n+1}(z;\lambda) = -q_{-n+1}(z;\lambda)$$

solutions of $\mathrm{P_{II}}$ for $\alpha = n - 1/2$, $n \leq 0$.

Airy Solutions of P_{II}



Zeros (blue) and poles (red) of $q_3(z;0)$.



Zeros (blue) and poles (red) of $q_3(z;\infty)$.

Rational Solutions of $\mathrm{P_{II}}$

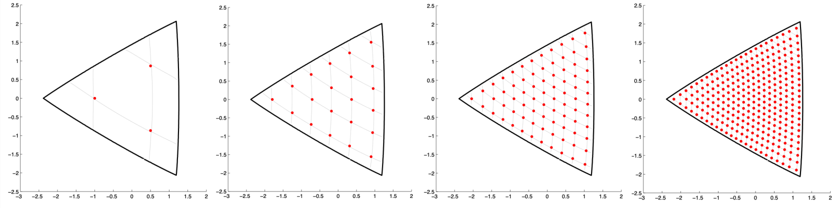
When $\alpha = n \geq 1$, $\mathrm{P_{II}}$ has a solution

$$u_n(z) = -\frac{d}{dz}\log\left(\frac{\mathcal{Q}_{n-1}(z)}{\mathcal{Q}_n(z)}\right),$$

where $\mathcal{Q}_n(z)$ are Vorob'ev-Yablonski polynomials defined via

$$\mathcal{Q}_{n+1}(z)\mathcal{Q}_n(z) = z\mathcal{Q}_n^2(z) - 4\left[\mathcal{Q}_n''(z)\mathcal{Q}_n(z) - \mathcal{Q}_n'(z)^2\right],$$

with $\mathcal{Q}_0(z) = 1$ and $\mathcal{Q}_1(z) = z$.



Zeros of $\mathcal{Q}_n(n^{2/3}z)$ for $n = 2, 6, 12, 24$.

Figure from M. Bertola and Th. Bothner, Zeros of large degree Vorob'ev-Yablonski polynomials via a Hankel determinant identity.

Rational Solutions of $\mathrm{P_{III}}$

When $\alpha = 4(m+n)$, $\beta = 4(m-n)$, $\gamma = -\delta = 4$, $\mathrm{P_{III}}$ has a solution

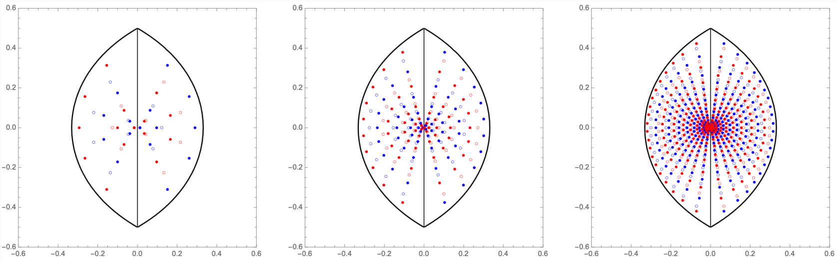
$$u_n(z;m) = \frac{s_n(z;m-1)s_{n-1}(z;m)}{s_n(z;m)s_{n-1}(z;m-1)},$$

where Umemura polynomials $s_n(z;m)$ are defined via

$$s_{n+1}(z;m) = \frac{(4z+2m+1)s_n^2(z) - s_n(z)s_n'(z) - z(s_n(z)s_n''(z) - s_n'(z)^2)}{2s_{n-1}(z;m)}$$

with $s_{-1}(z;m) = s_0(z;m) \equiv 1$.

Rational Solutions of P_{III}



Poles (red) and zeros (blue) of $u_n(nz;0)$ for $n = 5, 10, 20$.

Figure from Th. Bothner, P. Miller, and Y. Sheng, Rational solutions of the Painlevé-III equation.

Algebraic Solutions of $\mathrm{P_{III}}$

When $\alpha = 1$, $\beta = 2n$, $\gamma = 0$, and $\delta = -1$, $\mathrm{P_{III}}$ has a solution

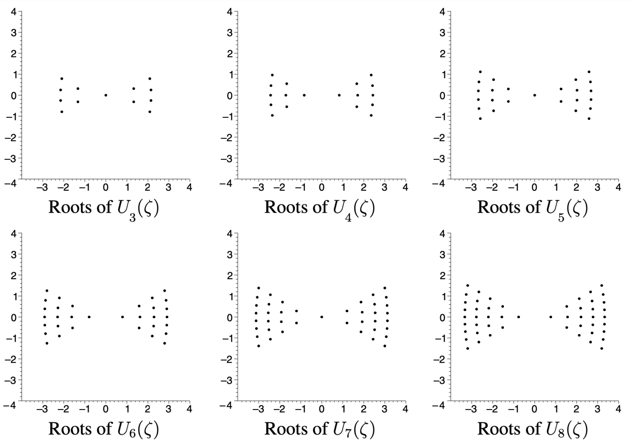
$$u_n(z) = \left(\frac{2}{3}\right)^{1/2} \frac{R_{n-1}(\zeta)R_{n+1}(\zeta)}{R_n^2(\zeta)}, \quad z = \left(\frac{2}{3}\right)^{3/2} \zeta^3,$$

where Ohyama polynomials $R_n(\zeta)$ are defined via

$$2\zeta R_{n+1}(\zeta)R_{n-1}(\zeta) = -R_n(\zeta)R_n''(\zeta) + R_n'(\zeta)^2 - \frac{1}{\zeta}R_n(\zeta)R_n'(\zeta) + 2(\zeta^2 - n)R_n^2(\zeta)$$

with $R_0(\zeta) = 1$ and $R_1(\zeta) = \zeta^2$.

Algebraic Solutions of P_{III}



Here $U_{2n}(\zeta) = \zeta^{-3n} R_{2n}(\zeta)$ and $\zeta^{-3n-1} R_{2n+1}(\zeta)$.

Figure from P. Clarkson, The third Painlevé equation and associated special polynomials.

Hankel Determinantal Representations

Hankel determinantal representation of special solutions of P_{IV} involving parabolic cylinder functions has been obtained by A. Barhoumi.

Hankel determinantal representation of Ohyama polynomials ($\mathrm{P}_{\mathrm{III}}$) has been very recently obtained by J. Harrow (not yet published).

Hankel determinantal representation of Vorob′ev-Yablonski polynomials (P_{II}) has been obtained by Bertola and Bothner.

It was shown by Okamoto and Flaschka-Newell that for the Airy solutions of P_{II} it holds that

$$q_n(z;\lambda) := \frac{d}{dz}\log\left(\frac{\tau_{n-1}(z;\lambda)}{\tau_n(z;\lambda)}\right),$$

where $\lambda = C_1/C_2$, $\tau_0(z;\lambda) = 1$ and for $n \geq 1$

$$\tau_n(z;\lambda) = \det\left[\frac{d^{j+k}}{dz^{j+k}}\left(C_1\mathrm{Ai}\left(-2^{-1/3}z\right) + C_2\mathrm{Bi}\left(-2^{-1/3}z\right)\right)\right]_{j,k=0}^{n-1}.$$

In the 1950's Wigner proposed to model the local statistical behavior of the resonance levels off heavy nuclei by the local statistical behavior of the eigenvalues of a large random matrix.

Let $\mathcal{H}_N$ be the space of $N \times N$ Hermitian matrices $\mathcal{M} = [\mathcal{M}_{ij}]$. Place a probability distribution

$$ce^{-F(\mathcal{M})} d\mathcal{M} = ce^{-F(\mathcal{M})} \prod d\mathcal{M}_{ii} \prod_{i<j} d\mathcal{M}_{ij}^R \mathcal{M}_{ij}^I$$

on $\mathcal{H}_N$, where $\mathcal{M}_{ij} = \mathcal{M}_{ij}^R + \mathrm{i}\mathcal{M}_{ij}^I$ and c is a normalizing constant.

Physically, since probabilities should not depend on the observer, it is required

$$F(\mathcal{M}) = F(\mathcal{U}\mathcal{M}\mathcal{U}^*)$$

for any unitary matrix $\mathcal{U}$ (it holds that $d(\mathcal{U}\mathcal{M}\mathcal{U}^*) = d\mathcal{M}$).

Gaussian Unitary Ensemble

Gaussian Unitary Ensemble is given by

$$F(\mathcal{M}) = \frac{1}{2}\operatorname{tr}\mathcal{M}^2 \quad \left(F_\beta(\mathcal{M}) = \frac{\beta}{4}\operatorname{tr}\mathcal{M}^2\right).$$

(G Orthogonal E is $\beta = 1$ and G Simplectic E is $\beta = 4$)

If f is a function such that $f(\mathcal{M}) = f(\mathcal{U}\mathcal{M}\mathcal{U}^*)$, then

$$f(\mathcal{M}) = f(\lambda) = f(\lambda_1, \lambda_2, \ldots, \lambda_N),$$

where λ_i are the eigenvalues of M. In this case "unitary" variables in $d\mathcal{M}$ can be integrated out and

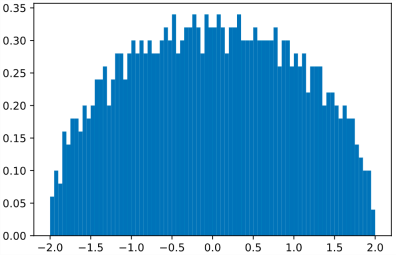
$$\int f(\mathcal{M}) e^{-\frac{1}{2}\operatorname{tr}\mathcal{M}^2} d\mathcal{M} \text{ “}=\text{” } \int_{\mathbb{R}^N} f(\lambda) e^{-\frac{1}{2}\sum_i \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 d^N\lambda.$$

The eigenvalues spread out on the whole real line. Therefore, it is more convenient to study the distribution of the scaled eigenvalues $x_i = \lambda_i/\sqrt{N}$:

$$\frac{1}{Z_N}\int_{\mathbb{R}^N} e^{-\frac{N}{2}\sum_i x_i^2} \prod_{i<j}(x_i - x_j)^2 d^N x,$$

where Z_N is the normalizing constant also known as a partition function.

Scaled eigenvalues converge to Wigner's semi-circle law: $\frac{1}{2\pi}\sqrt{4-x^2}dx$.



$N = 1000$

Gaussian Unitary Ensemble

Theorem (Tracy-Widom)

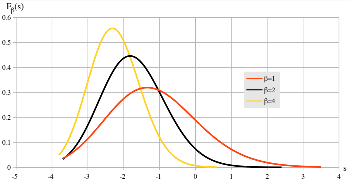
Let $x_{\max}(N)$ be the largest eigenvalue of an $N \times N$ scaled GUE matrix. Then

$$\lim_{N\to\infty} \text{Prob}\left((x_{\max}(N) - 2)N^{2/3} \leq s \right) = F_2(s),$$

where $F_2(s)$ is the Tracy-Widom distribution

$$F_2(s) = \exp\left\{ - \int_s^\infty (x - s) y^2(x) dx \right\}$$

and $y(x)$ is the Hastings-McLeod solution of $\mathrm{P_{II}}$.



https://en.wikipedia.org/wiki/Tracy-Widom_distribution

Quartic Perturbation

Consider the partition function of the quartic perturbation of GUE:

$$Z_N(t) = \int_{\mathcal{H}_N} e^{-\frac{1}{2}\operatorname{tr}\mathcal{M}^2 - \frac{t}{N}\operatorname{tr}\mathcal{M}^4} d\mathcal{M}.$$

Then it holds that

$$\frac{Z_N(-t)}{Z_N(0)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{N}\right)^N \left\langle \left(\operatorname{tr}\mathcal{M}^4\right)^n \right\rangle,$$

where

$$\langle f \rangle = Z_N^{-1}(0) \int f(\mathcal{M}) e^{-\frac{1}{2}\operatorname{tr}\mathcal{M}^2} d\mathcal{M}.$$

Wick's Calculus

$$\langle \operatorname{tr} \mathcal{M}^4 \rangle = \left\langle \sum_{i,j,k,l=1}^{N} \mathcal{M}_{ij}\mathcal{M}_{jk}\mathcal{M}_{kl}\mathcal{M}_{li} \right\rangle .$$

Wick's Theorem

It holds that

$$\begin{aligned}\langle \mathcal{M}_{ij}\mathcal{M}_{jk}\mathcal{M}_{kl}\mathcal{M}_{li} \rangle &= \langle \mathcal{M}_{ij}\mathcal{M}_{jk} \rangle \langle \mathcal{M}_{kl}\mathcal{M}_{li} \rangle + \\ &\langle \mathcal{M}_{ij}\mathcal{M}_{kl} \rangle \langle \mathcal{M}_{jk}\mathcal{M}_{li} \rangle + \langle \mathcal{M}_{ij}\mathcal{M}_{li} \rangle \langle \mathcal{M}_{jk}\mathcal{M}_{kl} \rangle .\end{aligned}$$

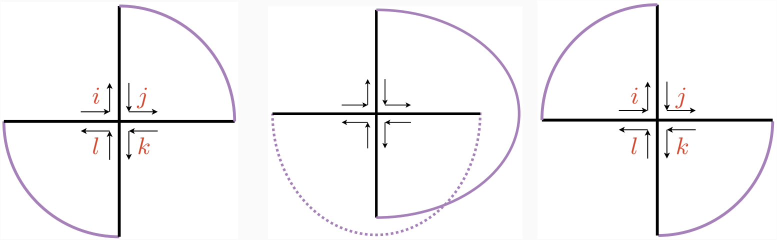
It also holds that

$$\langle \mathcal{M}_{ij}\mathcal{M}_{kl} \rangle = \delta_{il}\delta_{jk}.$$

Hence,

$$\langle \operatorname{tr} \mathcal{M}^4 \rangle = \sum_{i,j,k,l=1}^{N} (\delta_{ik} + \delta_{il}\delta_{jk}\delta_{ij} + \delta_{jl}) = N^3 + N + N^3.$$

The terms in Wick's theorem can be represented by Feynman's diagrams with one 4-valent vertex: $\mathcal{M}_{ij}\mathcal{M}_{kl}$ gives rise to $i \to l$ and $k \to j$



$$\langle \mathcal{M}_{ij}\mathcal{M}_{jk}\rangle \langle \mathcal{M}_{kl}\mathcal{M}_{li}\rangle + \langle \mathcal{M}_{ij}\mathcal{M}_{kl}\rangle \langle \mathcal{M}_{jk}\mathcal{M}_{li}\rangle + \langle \mathcal{M}_{ij}\mathcal{M}_{li}\rangle \langle \mathcal{M}_{jk}\mathcal{M}_{kl}\rangle$$

The first and last diagrams are planar with 3 faces and the second one is toroidal with 1 face.

In general, $\left(\operatorname{tr}\mathcal{M}^4\right)^n$ will correspond to a diagram with n 4-valent vertices.

Quartic Perturbation

"Theorem" (Bessis-Itzykson-Zuber)

Given the partition function

$$Z_N(t) = \int_{\mathcal{H}_N} e^{-\frac{1}{2}\operatorname{tr}\mathcal{M}^2 - \frac{t}{N}\operatorname{tr}\mathcal{M}^4} d\mathcal{M},$$

it holds that

$$\log\left(\frac{Z_N(-t)}{Z_N(0)}\right) \sim \sum_g e_g(t) N^{2-2g},$$

where $e_g(t)$ is analytic around the origin:

$$e_g(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!}\kappa_g(n),$$

and $\kappa_g(n)$ is the number of 4-valent connected graphs with n vertices on a Riemann surface of genus g.

Consider a more general Hermitian ensemble given by the distribution

$$\frac{1}{Z_N(\mathfrak{t})} \int_{\mathbb{R}^N} e^{-N \sum_i V(x_i;\mathfrak{t})} \prod_{i<j} (x_i - x_j)^2 d^N x,$$

where

$$V(x;\mathfrak{t}) = \frac{1}{2}x^2 + \sum_{i=1}^{2\nu} t_i x^i, \quad t_{2\nu} > 0,$$

and $\mathfrak{t} = (t_1, t_2, \ldots, t_{2\nu})$.

Theorem (Ercolani-McLaughlin)

Let $\mathfrak{t}$ be such that $|\mathfrak{t}| \leq T$ and $t_{2\nu} > \gamma \sum_{i=1}^{2n-1} t_i$, $T, \gamma > 0$. Then

$$\log \left(\frac{Z_N(\mathfrak{t})}{Z_N(\mathfrak{o})} \right) \sim \sum_{g=0}^{\infty} e_g(\mathfrak{t}) N^{2-2g},$$

where each $e_g(\mathfrak{t})$ is analytic in $\mathfrak{t}$ around $\mathfrak{o}$. Moreover,

$$e_g(-\mathfrak{t}) = \sum_{n_j \geq 1} \frac{t_1^{n_1} t_2^{n_2} \cdots t_{2\nu}^{n_{2\nu}}}{n_1! \, n_2! \cdots n_{2\nu}!} \kappa_g(n_1, n_2, \ldots, n_{2\nu}),$$

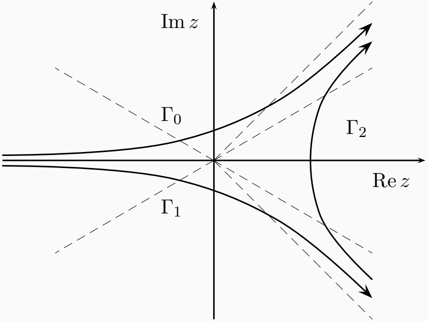
where $\kappa_g(n_1, n_2, \ldots, n_{2\nu})$ is the number of connected graphs with n_j j-valent vertices on a Riemann surface of genus g.

Complex Cubic Ensemble

Consider the partition function for the complex cubic ensemble:

$$Z_N(u;\alpha) = \int_{\Gamma^N} e^{-N\sum_i \left(\frac{z_i^2}{2} - u z_i^3\right)} \prod_{i<j} (z_i - z_j)^2 d^N z,$$

where $\Gamma = \alpha\Gamma_0 + (1-\alpha)\Gamma_1$, $\alpha \in \mathbb{C}$.



Theorem (Bleher-Deaño)

Fix $\alpha \in \mathbb{C}$. For $0 \leq u < 3^{1/4}/18$, $F_N(u;\alpha)$ admits an asymptotic expansion

$$\log \frac{Z_N(u;\alpha)}{Z_N(0;\alpha)} \sim \sum_{g=0}^{\infty} F^{(2g)}(u) N^{2-2g}.$$

The functions $F^{(2g)}(u)$ do not depend on α and admit an analytic continuation into $|u| < 3^{1/4}/18$. If

$$F^{(2g)}(u) = \sum_{j=1}^{\infty} f_{2j}^{(2g)} \frac{u^{2j}}{(2j)!},$$

then $f_{2j}^{(2g)}$ counts the number of 3-valent connected graphs with $2j$ vertices on a Riemann surface of genus g.

Complex Cubic Ensemble

After a substitution

$$z_j = (3u)^{-1/3} s_j + \frac{1}{6u}, \quad t = \frac{1}{4(3u)^{4/3}},$$

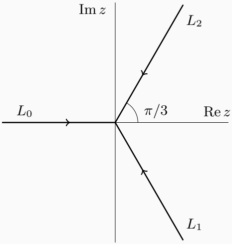
we get that

$$Z_N(u;\alpha) = (4t)^{N^2/4} e^{-2t^{3/2}N/3} Z_N(t),$$

where

$$Z_N(t) = \int_{\Gamma_\alpha^N} e^{-N\sum_i V(s_i;t)} \prod_{i<j} (s_i - s_j)^2 d^N s$$

$\Gamma_\alpha = L_0 - (1-\alpha)L_1 - \alpha L_2$ and $V(s;t) = -s^3/3 + ts$.



Let us now also reparametrize Γ_α as $\Gamma(\lambda)$, $\lambda \in \overline{\mathbb{C}}$, where

$$\Gamma(\lambda) = \frac{\lambda}{\pi} L_0 + \left(-\frac{\lambda}{2\pi} + \frac{1}{2\pi \mathrm{i}} \right) L_1 + \left(-\frac{\lambda}{2\pi} - \frac{1}{2\pi \mathrm{i}} \right) L_2$$

for $|\lambda| < \infty$ and $\Gamma(\infty) = \frac{1}{\pi} L_0 - \frac{1}{2\pi} L_1 - \frac{1}{2\pi} L_2$. Let

$$D_n(t; N, \lambda) = \det \left[\int_{\Gamma(\lambda)} s^{i+j} e^{-NV(s;t)} ds \right]_{i,j=0}^{n}.$$

Then

$$Z_N(t) = N! D_{N-1}(t; N, \lambda).$$

Connection to $\mathrm{P_{II}}$

Recall that we were interested in special solutions of $\mathrm{P_{II}}$ given by

$$q_n(z;\lambda) := \frac{d}{dz}\log\left(\frac{\tau_{n-1}(z;\lambda)}{\tau_n(z;\lambda)}\right)$$

where, as before, $\lambda = C_1/C_2$ and

$$\tau_{n+1}(z;\lambda) = \det\left[\frac{d^{j+k}}{dz^{j+k}}\left(C_1\mathrm{Ai}\left(-2^{-1/3}z\right) + C_2\mathrm{Bi}\left(-2^{-1/3}z\right)\right)\right]_{j,k=0}^{n}.$$

Recall also that

$$\mathrm{Ai}(z) + \lambda\mathrm{Bi}(z) = \left(\frac{1}{2\pi\mathrm{i}}\int_{L_1-L_2} + \frac{\lambda}{2\pi}\int_{L_0-L_1} + \frac{\lambda}{2\pi}\int_{L_0-L_2}\right)e^{-V(s;z)}ds.$$

Thus, it holds that

$$\tau_{n+1}(z;\lambda) = \frac{N^{N(N+1)/3}}{2^{(N-1)N/3}}D_n(-t;N,\lambda) \quad z = \left(\sqrt{2}N\right)^{2/3}t,$$

Orthogonal Polynomials

Let $P_n(s) = P_n(s; t, N, \lambda)$ be a polynomial of degree at most n such that

$$\int_{\Gamma(\lambda)} s^k P_n(s) e^{-NV(s;t)} ds = 0, \quad k = \overline{0, n-1}.$$

If $D_{n-1}(t; N, \lambda) \neq 0$, then $\deg P_n = n$. If $D_n(t; N, \lambda) D_{n-1}(t; N, \lambda) \neq 0$, then

$$sP_n(s) = P_{n+1}(s) + \beta_n(t; N, \lambda) P_n(s) + \gamma_n^2(t; N, \lambda) P_{n-1}(s).$$

Connection

Theorem (Barhoumi-Bleher-Deaño-Ya.)

$$q_n(z;\lambda) = -(N/2)^{1/3}\beta_{n-1}\left(-\left(\sqrt{2}N\right)^{-2/3}z; N, \lambda\right).$$

In particular, poles of $q_n(z;\lambda)$ correspond to values of t for which

$$D_n(t;N,\lambda) = 0 \quad \text{or} \quad D_{n-1}(t;N,\lambda) = 0.$$

That is, values of t for which

$$\deg P_n = n-1 \quad \text{or} \quad \deg P_{n+1} = n.$$

Moreover,

$$p_n(z;\lambda) = -2(N/2)^{2/3}\gamma_{n-1}^2\left(-\left(\sqrt{2}N\right)^{-2/3}z; N, \lambda\right)$$

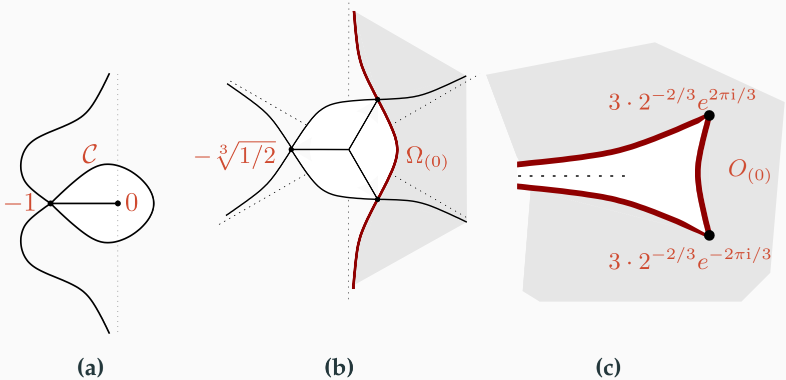
gives an associated special solution of $\mathrm{P}_{\mathrm{XXXIV}}$.

Heuristics

Heuristics

If the attracting set of zeros of P_n consists of a single Jordan arc, all the zeros of P_n remain bounded; otherwise P_n will have a zero exhibiting a spurious behavior.

Phase Space

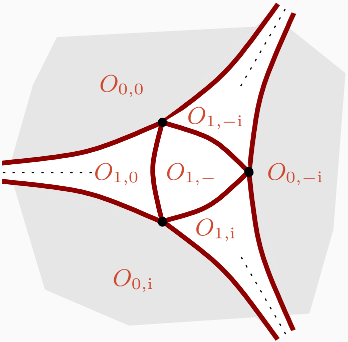


(a) $\mathcal{C}$ is the critical graph of the quadratic differential $-\left(1+\frac{1}{s}\right)^3 ds^2$.

(b) $\Delta := \left\{x : \; 2x^3 \in \mathcal{C}\right\}$.

(c) $O_{(0)} = t(\Omega_{(0)})$, $x^3 - tx - 1 = 0$.

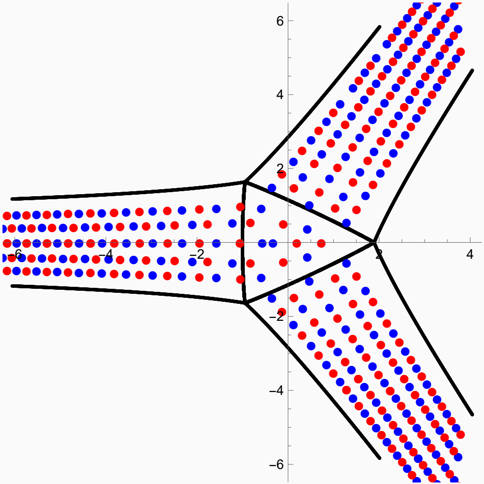
Phase Space



Theorem (Barhoumi-Bleher-Deaño-Ya.)

- for $t \in O_0$ the attracting set is a single analytic Jordan arc;
- for $t \in O_{1,-}$ the attracting set consists of three analytic Jordan arcs with a common endpoint;
- for $t \in O_1 \setminus O_{1,-}$ attracting set consists of two analytic Jordan arcs with no endpoints in common.

Phase Space



Poles (red) and zeros (blue) of $q_3(-(3\sqrt{2})^{2/3}z;\infty)$ and the curves ∂O_1 (black).

Scaling Limit in the One-cut Region

Theorem (Barhoumi-Bleher-Deaño-Ya.)

When $\lambda = 0$ (pure Airy solution),

$$(2/n)^{1/3} q_n \left(-(\sqrt{2}n)^{2/3} t; 0 \right) = -x_0(t) + \mathcal{O}\left(n^{-1}\right)$$

and

$$(2/n)^{2/3} p_n \left(-(\sqrt{2}n)^{2/3} t; 0 \right) = 1/x_0(t) + \mathcal{O}\left(n^{-2}\right)$$

where $x_0(O_{(0)}) = \Omega_{(0)}$, $x^3 - tx - 1 = 0$.

Theorem (Barhoumi-Bleher-Deaño-Ya.)

When $\lambda = 0$ (pure Airy solution),

$$\gamma_n^2(t; n, 0) = S(t)\theta_{n,1}(\infty; t)\theta_{n,2}(\infty; t) + \mathcal{O}\left(n^{-1}\right)$$

where $S(t)$ is an explicit functions of the endpoints of the attracting set of zeros of the orthogonal polynomials and $\theta_{n,i}(z; t)$ are the pull-backs from one of the sheets of a two-sheeted Riemann surface associated to the attracting set, one having a floating pole and one having a floating zero depending on n, t.