ON RATES IN RATIONAL AND MEROMORPHIC APPROXIMATION

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As we have learned after completing this note, its main result has already been proven by Vasiliy Prokhorov [20].

Abstract. In this note we study the relation between the limit superior and the limit inferior of the n-th root error in rational approximation of degree n to holomorphic functions, in the supremum norm on a continuum contained in the domain of holomorphy. We stress links with meromorphic approximation, initially introduced by Parfënov in this context.

1. Introduction and main results

Let A be a closed set in the complex plane $\mathbb{C}$ and $\mathcal{H}(A)$ the space of functions holomorphic in a (variable) neighborhood of A. One of the central questions of complex approximation theory is to investigate the behavior of

$$\rho_n(f,A) := \inf_{\mathcal{R}_n(\mathbb{C}\setminus A)} \|f-r\|_A,$$

where $f \in \mathcal{H}(A)$, $\|\cdot\|_A$ is the supremum norm on A, and $\mathcal{R}_n(\Omega)$ is the class of rational functions of type (n,n) with all their poles in Ω; specifically, letting $\mathcal{P}_n$ be the space of algebraic polynomials of degree at most n and $\mathcal{M}_n(\Omega)$ the set of monic polynomials of degree n with all their zeros in Ω, it holds that $\mathcal{R}_n(\Omega) := \mathcal{P}_n\mathcal{M}_n^{-1}(\Omega)$.

It was shown by Walsh [27, 1] using techniques of rational interpolation that

$$\limsup_{n\to\infty} \rho_n^{1/n}(f,A) \leqslant \exp\left\{-\frac{1}{\operatorname{cap}(A,K)}\right\}, \tag{1}$$

where K is any closed set in the complement of which f is holomorphic and $\operatorname{cap}(A,K)$ is the capacity of the condenser (A,K); for notions of potential theory, we refer the reader to the excellent monographs [21, 22, 12]. In general, it is known that Walsh's inequality cannot be improved [13]. However, it was conjectured by Gonchar [11] and proved in the case of continua with connected complement by Parfënov [16], and for arbitrary compact sets by Prokhorov [19], that

$$\liminf_{n\to\infty} \rho_n^{1/n}(f,A) \leqslant \exp\left\{-\frac{2}{\operatorname{cap}(A,K)}\right\}. \tag{2}$$

Hence, $\rho_n(f,A)$ has no limit in general as $n\to\infty$, and when the limit exists it cannot exceed the infimum of the right-hand side of (2) over all compact K outside of which f is analytic and single-valued. The limit indeed exists for certain classes of functions: for instance, if f belongs to the class $\mathcal{S}(A)$ of functions in $\mathcal{H}(A)$ admitting a multiply sheeted analytic continuation to every point of the complement of a closed, polar set (i.e., if f can be continued over the extended complex plane $\overline{\mathbb{C}}$ into a multi-valued function with a closed, polar singular set), then it follows from the work of Stahl [23, 24, 25] and of Gonchar and Rakhmanov [9, Theorem 1′] that

$$\lim_{n\to\infty} \rho_n^{1/n}(f,A) = \inf_K \exp\left\{-\frac{2}{\operatorname{cap}(A,K)}\right\},$$

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where the infimum is taken over all compact sets K such that f admits an analytic and single-valued continuation to their complement. The existence of a compact set K_f minimizing $\text{cap}(A, K)$ under this constraint, and its uniqueness up to addition or removal of a set of capacity zero, follow from the work in [23, 24] as soon as $f \in \mathcal{H}(A)$.

The goal of this note is to further elucidate the connection between the behavior of the limit superior in (1) and the limit inferior in (2) when no particular assumption is made on continuability properties of f. More precisely, we wish to prove the following theorem.

Theorem 1. *Let $f \in \mathcal{H}(A)$, where A is a continuum. If one defines the number $b > 0$ by the limiting relation*

$$\limsup_{n\to\infty} \rho_n^{1/n}(f, A) = \exp\left\{-\frac{b}{\text{cap}(A, K)}\right\},$$

where K is a closed set disjoint from A to the complement of which f admits an analytic and single-valued continuation, then $b \geqslant 1$ and

$$\liminf_{n\to\infty} \rho_n^{1/n}(f, A) \leqslant \exp\left\{-\frac{\max\{b, b/(2-2/b)\}}{\text{cap}(A, K)}\right\}. \tag{3}$$

Of necessity $b \geqslant 1$, by (1). However, let us emphasize that our approach provides an independent proof of Walsh's result. The theorem asserts, in a quantitative manner, that if along some subsequence of degrees the n-th-root asymptotic error rate is larger than Parfënov's upper bound for the liminf given by the right-hand side of (2) with $K = K_f$, then along some other subsequence the corresponding rate must be smaller than this bound. It is so, for if $b \in [1, 2)$ then $b/(2 - 2/b) \in (2, \infty]$ and the endpoint cases are:

$$b = 2 \;\Leftrightarrow\; \frac{b}{2 - 2/b} = 2 \quad \text{and} \quad b = 1 \;\Leftrightarrow\; \frac{b}{2 - 2/b} = \infty. \tag{4}$$

In particular, Theorem 1 generalizes the observation, made in [19], that if along some subsequence of degrees the best rational approximants have worst possible asymptotic n-th root error in rational approximation (given by the right-hand side of (1) with $K = K_f$), then along some other subsequence such approximants will converge superexponentially fast with the degree. It may also be worth stating the following consequence of Theorem 1, an analog of which was shown in [9] for multipoint Padé approximants to functions from $\mathcal{S}(A)$, though in the context of convergence in capacity.

Corollary 2. *With the notation and assumptions of Theorem 1, if (2) is an equality, then $\rho_n^{1/n}(f, A)$ does have a limit when $n \to \infty$.*

Proof. This follows readily from (3) and (4). □

Our proof elaborates on the approach taken in [16], trading rational approximation for meromorphic approximation, and exploiting the connection of the latter with the spectral analysis of Hankel operators as stressed in the so-called AAK-theory (for Adamjan-Arov-Krein); see [17, Chapter 4]. Our main contribution here is to analyze more thoroughly than [16] the Horn-Weyl inequalities for the Hankel operators with symbol f, and to derive from them the quantitative rule set forth in Theorem 1. Another, lesser contribution is to clarify issues of existence and uniqueness of best meromorphic approximants on ∂A to functions in $\mathcal{H}(A)$, in the case where ∂A is a non-rectifiable Jordan curve.

2. Meromorphic approximants

We set $D := \overline{\mathbb{C}} \setminus A$, which is a simply connected domain since A is a continuum. In this case, one has $\text{cap}(A, K) = \text{cap}_D(K)$ for any compact set $K \subset D$, where $\text{cap}_D(\cdot)$ is the Greenian capacity of K as a subset of D, see [22, 12]. Following the original approach of Parfënov, we proceed by enlarging the class of approximants from rational to meromorphic ones. However, we do it only under the additional assumption that $T := \partial A$ is a Jordan curve. We shall keep this assumption in force all the way up to the proof of Theorem 1. Moreover, since composition with a Möbius transformation preserves rational functions and their degree, we shall assume from now on that A is unbounded and $T := \partial A = \partial D$ is compact; i.e., that the point at infinity lies interior to A.

Let $H^\infty(D)$ denote the space of bounded analytic functions on D and $\mathcal{A}(D)$ the subspace of those extending continuously to $T(= \partial D)$. Further, let us put

$$\begin{cases} H_n^\infty(D) & := \{h/q : \ h \in H^\infty(D),\ q \in \mathcal{M}_n(D)\}, \\ \mathcal{A}_n(D) & := \{h/q : \ h \in \mathcal{A}(D),\ q \in \mathcal{M}_n(D)\}. \end{cases}$$

That is, $H_n^\infty(D)$ is the set of meromorphic function with at most n poles in D that are bounded near T, and $\mathcal{A}_n(D)$ is the subset of those extending continuously to T. Note that $\mathcal{R}_n(D) \subset \mathcal{A}_n(D)$.

Assume first that $T = \mathbb{T}$, the unit circle. The meromorphic approximation (Nehari-Takagi) problem we are interested in is the following one: *given $f \in L^\infty(\mathbb{T})$, find $M_n^\infty \in H_n^\infty(\mathbb{D})$ such that*[1]

$$\|f - M_n^\infty\|_{\mathbb{T}} = \inf_{M \in H_n^\infty(\mathbb{D})} \|f - M\|_{\mathbb{T}}. \tag{5}$$

It is known that M_n^∞ always exists and that it is unique when f lies in $C(\mathbb{T}) + H^\infty(\mathbb{D})$, see [17, Chapter 4]. Moreover, if f is Dini-continuous on $\mathbb{T}$, then M_n^∞ is continuous on $\mathbb{T}$; indeed, if we write $M_n^\infty = r_n + g$ where $r_n \in \mathcal{R}_n(\mathbb{D})$ and $g \in H^\infty(\mathbb{D})$, then g must be the best Nehari approximant to $f - r_n$ and the latter is Dini-continuous on $\mathbb{T}$, so the claim follows from [4].

When T is rectifiable, each $h \in H^\infty(D)$ has a non-tangential limit almost everywhere on T with respect to arclength, that we still call h, and putting $\|\cdot\|_T$ for the essential supremum norm on T (with respect to arclength) it holds that $\|h\|_T = \|h\|_D$ [5, Theorems 10.3 & 10.5]. Thus, in this case, one can readily replace $\mathbb{D}$ by D in (5) and carry over to T all properties of best meromorphic approximants on $\mathbb{T}$ by conformal mapping.

When T is a non-rectifiable Jordan curve, however, limiting values on T of $H^\infty(D)$-functions generally exist at sectorially accessible points only, and such points may reduce to a set of zero linear measure [14]. In this case, the very existence of best meromorphic approximants depends on the analyticity of the approximated function across T.

Lemma 3. *Let T, A and D be as above. Given $f \in \mathcal{H}(A)$, to each integer n there exists a unique $M_n^\infty \in \mathcal{A}_n(D)$ such that*

$$\|f - M_n^\infty\|_T = \inf_{M \in \mathcal{A}_n(D)} \|f - M\|_T. \tag{6}$$

If T is rectifiable, $\mathcal{A}_n(D)$ can be replaced by $H_n^\infty(D)$ in (6) without changing M_n^∞.

The proof of the above lemma can be found in [2]. However, since it is short, we do present it for completeness of the exposition. Before we start, let us recall some definitions and make pieces of notation. We denote by $L^2(\mathbb{T})$ the space of square integrable functions on $\mathbb{T}$, and let $H^2 \subset L^2(\mathbb{T})$ be the Hardy space of functions whose Fourier coefficients with negative index do vanish. It is known that H^2 can be identified with (non-tangential limits a.e. on $\mathbb{T}$ of) analytic functions in $\mathbb{D}$ whose L^2-means on circles centered at the origin are uniformly bounded, see [5, Theorem 3.4]. We set $H^2_- := L^2(\mathbb{T}) \ominus H^2$ to be the orthogonal complement of H^2, which is the Hardy space of L^2-functions whose Fourier coefficients with nonnegative index are equal to zero. The latter can be identified with functions analytic in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ that vanish at infinity, and whose L^2-means with respect to normalized arclength on circles centered at the origin are uniformly bounded. We let $\mathbb{P}_- : L^2(\mathbb{T}) \to H^2_-$ and

[1] If $n = 0$, (5) reduces to the question of best analytic approximation of bounded functions on the unit circle by elements of $H^\infty(\mathbb{D})$, which is the so-called *Nehari problem* named after [15] (that deals with an equivalent issue).

$\mathbb{P}_+ : L^2(\mathbb{T}) \to H^2$ be the orthogonal projections. Observe that $\mathbb{P}_\pm$ admit standard integral representations via Cauchy integrals:

$$\mathbb{P}_+(f)(z) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}}\frac{f(\zeta)}{\zeta - z}d\zeta, \quad |z|<1; \quad \mathbb{P}_-(f)(z) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}}\frac{f(\zeta)}{z-\zeta}d\zeta, \quad |z|>1. \tag{7}$$

Proof of Lemma 3. Let $\phi : \mathbb{D} \to D$ be a conformal map. Since D is a Jordan domain, ϕ extends to a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{D}$ by Carathéodory theorem [18, Section 2]. If we pick a continuous function f on T, then $f \circ \phi$ is continuous on $\mathbb{T}$ and, in particular, it lies in $L^2(\mathbb{T})$. Set $F := \mathbb{P}_-(f\circ\phi)$ and $G := \mathbb{P}_+(f\circ\phi)$, so that $F \in H^2_-$ and $G \in H^2$. Since $\mathbb{P}_+ + \mathbb{P}_-$ is the identity operator on $L^2(\mathbb{T})$, it holds that

$$F(z) = (f\circ\phi)(z) - G(z), \quad \text{a.e. } z \in \mathbb{T}. \tag{8}$$

Now, if $f \in \mathcal{H}(A)$, then $f \circ \phi(z)$ is holomorphic in $r < |z| < 1$ and continuous in $r \leqslant |z| \leqslant 1$, for r close enough to 1. Hence, the right-hand side of (8) is holomorphic in $r < |z| < 1$ with uniformly bounded L^2-means on circles centered at the origin, while the left-hand side lies in H^2_- and both sides have the same non-tangential limit on $\mathbb{T}$. By an easy variant of Morera's theorem [7, Chapter II, Exercise 12], the function equal to $F(z)$ for $|z| > 1$ and to $(f\circ\phi)(z) - G(z)$ for $r < |z| < 1$ is holomorphic across $\mathbb{T}$, in particular F extends analytically across $\mathbb{T}$ and G extends continuously to $\overline{\mathbb{D}}$. Then, as mentioned after (5), the best meromorphic approximant $M_n^\infty \in H_n^\infty(\mathbb{D})$ to $f\circ\phi$ exists and is unique, moreover it is readily checked that M_n^∞ is equal to the sum of G (a member of $\mathcal{A}(\mathbb{D})$) and of the best approximant to F from $H_n^\infty(\mathbb{D})$, which lies in $\mathcal{A}_n(\mathbb{D})$ because F is analytic across $\mathbb{T}$ and therefore is Dini-continuous on $\mathbb{T}$; hence, we get that $M_n^\infty \in \mathcal{A}_n(\mathbb{D})$. If now $M \in \mathcal{A}_n(D)$, then $M\circ\phi \in \mathcal{A}_n(\mathbb{D})$ and

$$\|f - M\|_T = \|f\circ\phi - M\circ\phi\|_{\mathbb{T}} \geqslant \|f\circ\phi - M_n^\infty\|_{\mathbb{T}}$$

by definition of M_n^∞. As $M_n^\infty \circ \phi^{-1} \in \mathcal{A}_n(D)$, it is the unique best meromorphic approximant to f we are looking for. The previous argument also shows that, when $f \in \mathcal{H}(A)$, the best meromorphic approximant to $f\circ\phi$ necessarily belongs to $\mathcal{A}_n(\mathbb{D})$. If T is rectifiable then composition with ϕ is an isometric isomorphism $L^\infty(T) \to L^\infty(\mathbb{T})$ (understood with respect to the arclength measure), hence one can equivalently use $H_n^\infty(D)$ instead of $\mathcal{A}_n(D)$ in definition (6). □

The next theorem stands analogous to Theorem 1 for meromorphic approximants.

Theorem 4. *Let T, A and D be as above. Further, let $f \in \mathcal{H}(A)$ and the approximants $M_n^\infty \in \mathcal{A}_n(D)$ be as in Lemma 3. If a number $b \geqslant 0$ is such that*

$$\limsup_{n\to\infty}\|f - M_n^\infty\|_T^{1/n} \geqslant \exp\left\{\frac{-b}{\mathrm{cap}_D(K)}\right\}, \tag{9}$$

where $K \subset D$ is a closed set to the complement of which f admits an analytic and single-valued continuation, then $b \geqslant 1$[2] and

$$\liminf_{n\to\infty}\|f - M_n^\infty\|_T^{1/n} \leqslant \exp\left\{-\frac{\max\{b, b/(2-2/b)\}}{\mathrm{cap}_D(K)}\right\}. \tag{10}$$

As we shall see later, the proof of Theorem 4 quickly reduces to the case of the unit disk. Our argument will be a refinement of Parfënov's original proof of estimate (2) given in [16], that we review in the next section on our way to the proof of Theorem 4.

[2]Since the inclusion $\mathcal{R}_n(D) \subset \mathcal{A}_n(D)$ implies that $\|f - M_n^\infty\|_T \leqslant \rho_n(f, A)$, the conclusion $b \geqslant 1$ is a consequence of (1), but it also independently follows from our analysis.

3. Factorization of Hankel operators and n-th root singular values estimates

Parfënov's argument is based on AAK-theory and a factorization of the Hankel operator with symbol f that involves embedding operators. The latter have singular values whose n-th root asymptotics is known, and the desired estimate (2) then drops out of the Horn-Weyl inequalities when applied to this factorization. Again, we assume without loss of generality that $f(\infty) = 0$.

Recall the definition of Hardy spaces after the statement of Lemma 3. Given $h \in L^\infty(\mathbb{T})$, one defines the *Hankel operator with symbol* h to be

$$\Gamma_h : H^2 \to H^2_-, \qquad \Gamma_h(g) := \mathbb{P}_-(gh).$$

For a non-negative integer n, let $s_n(\Gamma_h)$ be the $(n+1)$-th singular number of the operator Γ_h; that is, $s_n(\Gamma_h) := \inf_{\operatorname{rank} R \leqslant n} \|\Gamma_h - R\|$, where the infimum is taken over all operators $R : H^2 \to H^2_-$ of rank at most n and $\|\cdot\|$ stands for the operator norm. Then, it holds that

$$\inf_{M \in H^\infty_n(\mathbb{D})} \|h - M\|_{\mathbb{T}} = s_n(\Gamma_h). \tag{11}$$

If, in addition, $h \in C(\mathbb{T}) + H^\infty(\mathbb{D})$, then Γ_h is compact so that $s_n^2(\Gamma_h)$ is the $(n+1)$-st eigenvalue of $\Gamma_h^*\Gamma_h$, when these are arranged in non-increasing order. Moreover in this case, the infimum is uniquely attained at $M = M_n^\infty \in H_n^\infty(\mathbb{D})$ and (11) becomes a pointwise equality:

$$|(h - M_n^\infty)(z)| = s_n(\Gamma_h) \quad \text{a.e. on} \quad \mathbb{T}. \tag{12}$$

Going back to the subject matter of Theorem 4, let L be a finite disjoint union of C^∞-smooth Jordan curves in $\mathbb{D}$, whose interior components cover the set K. Further, let L be oriented counterclockwise and denote its exterior by Ω. Cauchy's formula then yields that

$$f(z) = \int_L \frac{f(s)}{z-s}\frac{ds}{2\pi \mathrm{i}}, \qquad z \in \Omega.$$

Recall from (7) that the orthogonal projection $\mathbb{P}_-$ can be computed via a Cauchy integral on $\mathbb{T}$. Hence, it follows from Fubini's theorem and the residue formula that

$$(\Gamma_f g)(z) = \int_L \left(\int_{\mathbb{T}} \frac{g(\zeta)}{(z-\zeta)(\zeta - s)} \frac{d\zeta}{2\pi\mathrm{i}} \right) f(s) \frac{ds}{2\pi\mathrm{i}} = \int_L \frac{g(s) f(s)}{z-s}\frac{ds}{2\pi\mathrm{i}}, \qquad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}, \tag{13}$$

for any $g \in H^2$. This, of course, shows that $\Gamma_f g$ extends to an analytic function in Ω, and thus to an analytic function in $\overline{\mathbb{C}} \setminus K$ upon varying L.

Before we proceed with the factorization of Γ_f, recall that the Smirnov class $\mathcal{E}^2(\Omega)$ consists of all functions h holomorphic in Ω for which there exists an increasing sequence of domains Ω_n, with $\overline{\Omega}_n \subset \Omega$, such that $\partial\Omega_n$ is a finite union of rectifiable Jordan arcs and each compact subset of Ω is eventually contained in Ω_n for n large enough, while

$$\sup_{n \in \mathbb{N}} \|h\|_{L^2(\partial\Omega_n)} < \infty, \tag{14}$$

where $L^2(\partial\Omega_n)$ is meant with respect to arclength. The set $\mathcal{E}^2(\Omega)$ forms a vector space, although this may not be obvious at first glance, because in fact there is a single system $\{\Omega_n\}_{n\in\mathbb{N}}$ that works for all $h \in \mathcal{E}^2(\Omega)$ at the same time. The norm defined by (14) then makes $\mathcal{E}^2(\Omega)$ into a Hilbert space [5, Chapter 10]. We further denote by $\mathcal{E}^2_0(\Omega)$ the subspace of functions vanishing at infinity. Since the boundary $L = \partial\Omega$ consists of finitely many rectifiable Jordan curves, each $h \in \mathcal{E}^2(\Omega)$ has non-tangential limits a.e. with respect to arclength on L, and the boundary functions thus defined a.e. on L belong to $L^2(L)$. The boundary function characterizes $h \in \mathcal{E}^2(\Omega)$ completely, in that it cannot vanish on a set of positive arclength on L unless $h \equiv 0$. The $L^2(L)$-norm of its boundary function matches the norm (14) of h in $\mathcal{E}^2(\Omega)$. Consequently, we can identify $\mathcal{E}^2(\Omega)$ with a closed subspace of $L^2(L)$. Moreover, a function $h \in \mathcal{E}^2(\Omega)$ can be recovered from its boundary values on $\partial\Omega$ by a Cauchy integral, see [5, Section 10.5]. The correspondence just discussed makes it possible to regard $\mathcal{E}^2(\Omega)$ both as a Hilbert space of analytic functions in Ω and as a Hilbert space of functions on L. Finally, since L consists of smooth and therefore Ahlfors-regular Jordan curves, the Cauchy integral operator defines a bounded projection from $L^2(L)$ onto $\mathcal{E}_0^2(\Omega)$, see [5, Section 10.5] and [3, Corollary 6.8].

Using (13), we see that the Hankel operator Γ_f is a composition of four "elementary" operators, namely $\Gamma_f = B_1 B_2 B_3 B_4$, where

(15)
$$\begin{cases} B_4 : H^2 \to L^2(L) \text{ is the embedding operator obtained by restricting functions to } L; \\ B_3 : L^2(L) \to L^2(L) \text{ is the multiplication by } f; \\ B_2 : L^2(L) \to \mathcal{E}_0^2(\Omega) \text{ is the Cauchy projection onto the Smirnov class;} \\ B_1 : \mathcal{E}_0^2(\Omega) \to H^2_- \text{ is the embedding operator arising by restriction to } \overline{\mathbb{C}}\setminus\overline{\mathbb{D}}. \end{cases}$$

The operator B_3 is clearly bounded and the boundedness of B_2 follows from the discussion above. The operators B_1 and B_4 are immediately seen to be compact by a normal family argument. Further it is known, see [29], that their singular numbers must satisfy

$$\lim_{k\to\infty} s_k^{1/k}(B_1) = \lim_{k\to\infty} s_k^{1/k}(B_4) = \exp\left\{-\frac{1}{\operatorname{cap}_{\mathbb{D}}(K_L)}\right\}, \tag{16}$$

where K_L is the closure of the interior domain of L. Indeed, the second limit in (16) follows from [6, Chapter 7, Theorem 6.4], once it is recognized that the Green equilibrium density $\mu_{\mathbb{D},L}$ on L is smooth with non-vanishing density $d\mu_{\mathbb{D},L}(s)/|ds|$ with respect to arclength. To see this, observe that the equilibrium potential solves the Dirichlet problem in Ω with smooth (even constant) boundary data on $\mathbb{T}$ and L, hence it is smooth and so is its normal derivative which is precisely $d\mu_{\mathbb{D},L}(s)/|ds|$; since every point of L is a maximum place for the equilibrium potential, the density in question cannot vanish by the Hopf lemma. The conclusion can be verified also by methods presented in [22, Section IV.2] for the determination of equilibrium measures. The first limit in (16) is obtained by applying the same argument to the adjoint operator B_1^*, which is unitarily equivalent to the embedding operator $H^2_- \to L^2(L^*)$, where L^* is the reflection of L across $\mathbb{T}$.

It has been the pioneering observation of Parfënov that one can estimate the product of the first n singular values of the Hankel operator Γ_f by using (16) together with the Horn-Weyl inequalities:

$$\Pi_{k=0}^n s_k(EF) \leqslant \Pi_{k=0}^n s_k(E)\, \Pi_{k=0}^n s_k(F), \quad n \in \mathbb{N}, \tag{17}$$

which are valid for any bounded operators $E : \mathcal{H}_1 \to \mathcal{H}_2$ and $F : \mathcal{H}_2 \to \mathcal{H}_3$ between Hilbert spaces, see [8]. Indeed, using the obvious inequality $s_k(EF) \leqslant \min\{\|E\| s_k(F), \|F\| s_k(E)\}$, we deduce from (17) by putting $E = B_1 B_2$ and $F = B_3 B_4$ that

$$\Pi_{k=0}^n s_k(\Gamma_f) \leqslant \|B_2\|^{n+1} \|B_3\|^{n+1} \Pi_{k=0}^n s_k(B_1)\, \Pi_{k=0}^n s_k(B_4), \quad n \in \mathbb{N}. \tag{18}$$

Combination of inequalities (16) and (18) now yields that

$$\liminf_{k\to\infty} s_k^{1/k}(\Gamma_f) \leqslant \exp\left\{-\frac{2}{\operatorname{cap}_{\mathbb{D}}(K_L)}\right\}.$$

It is known that the Greenian capacity is right continuous on compact sets. That is, if we select a sequence $\{L_n\}$, where each L_n is a finite disjoint union of C^∞-smooth Jordan curves, such that $K = \cap_n K_{L_n}$, then

$$\operatorname{cap}_{\mathbb{D}}(K) = \lim_n \operatorname{cap}_{\mathbb{D}}(K_{L_n}). \tag{19}$$

Such a choice for the sequence $\{L_n\}$ is always possible due to Whitney's theorem (asserting that any compact set is the zero set of a non-negative C^∞-smooth function), see [26, Theorem VI.2] or [28, Theorem I], and to Sard's theorem [10, page 62] (asserting that almost every level line of a C^∞-smooth function is smooth, and thus will satisfy the conditions placed on L_n). Hence, we always can find L_n such that $\mathrm{cap}_{\mathbb{D}}(K_{L_n})$ (which is no less than $(\mathrm{cap}_{\mathbb{D}}(K))$) is in fact arbitrary close to $\mathrm{cap}_{\mathbb{D}}(K)$. Clearly, this yields that

$$\liminf_{k\to\infty} s_k^{1/k}(\Gamma_f) \leqslant \exp\left\{-\frac{2}{\mathrm{cap}_{\mathbb{D}}(K)}\right\},$$

which is precisely the result of Parfënov [16]. The improvement of this inequality stated in Theorem 4 rests on the following lemma, which refines the above analysis.

Lemma 5. *Let $\{a_k\}_{k\geqslant 1}$ and $\{b_k\}_{k\geqslant 1}$ be two sequences of non-negative real numbers, of which $\{a_k\}$ is non-increasing, that satisfy*

$$\Pi_{k=1}^n a_k \leqslant \Pi_{k=1}^n b_k \quad \text{for all} \quad n\in\mathbb{N}. \tag{20}$$

Assume for some $\rho\in(0,1)$ that

$$\limsup_{k\to\infty} b_k^{1/k} \leqslant \rho. \tag{21}$$

If λ is a non-negative real number such that

$$\limsup_{k\to\infty} a_k^{1/k} > \rho^{\lambda}, \tag{22}$$

then $1/2<\lambda$. Moreover, if $\lambda<1$, then

$$\liminf_{k\to\infty} a_k^{1/k} < \rho^{\,\lambda^2/(2\lambda-1)}. \tag{23}$$

Proof. By (21), there is to each $\varepsilon>0$ a constant C_ε such that

$$b_k \leqslant C_\varepsilon(\rho+\varepsilon)^k \quad \text{for all} \quad k\in\mathbb{N}. \tag{24}$$

Thus, we see from (20), (24), and the monotonicity of a_k that $a_k^k \leqslant C_\varepsilon^k(\rho+\varepsilon)^{k(k+1)/2}$. As $\varepsilon>0$ is arbitrary, the fact that $\lambda>1/2$ follows from the inequalities

$$\rho^\lambda < \limsup_{k\to\infty} a_k^{1/k} \leqslant \limsup_{k\to\infty} C_\epsilon^{1/k}(\rho+\varepsilon)^{(1+1/k)/2} = (\rho+\varepsilon)^{1/2}.$$

Suppose now that $\lambda<1$. If the lim inf in (23) is equal to zero, there is nothing more to prove. Otherwise, let $\alpha>\lambda$ be such that

$$\liminf_{k\to\infty} a_k^{1/k} > \rho^{\alpha}. \tag{25}$$

Since $a_k\neq 0$ for all k by monotonicity, this implies the existence of a strictly positive constant c_1 for which

$$a_k > c_1\rho^{\alpha k} \quad \text{for all} \quad k\in\mathbb{N}. \tag{26}$$

Likewise, by (22), there is an infinite subsequence $\{a_{k_n}\}_{n\geqslant 1}$ and a strictly positive constant c_2 such that

$$a_{k_n} > c_2\rho^{\lambda k_n} \quad \text{for all} \quad n\in\mathbb{N}. \tag{27}$$

Given a natural number $m>1$, put $\delta := (\alpha-\lambda)/(m-1)>0$ and consider the decreasing sequence of m positive real numbers defined by

$$s_j := \lambda+(m-j)\delta, \quad 1\leqslant j\leqslant m,$$

so that $s_m=\lambda$ and $s_1=\alpha$. Choose $n>1$ and define

$$\ell_j := \left\lceil \frac{\lambda k_n}{s_j}\right\rceil, \quad 1\leqslant j\leqslant m,$$

where $\lceil x\rceil$ indicates the smallest integer which is bigger than or equal to x. Note that $\ell_m=k_n$ and that $\ell_1\geqslant 1$. It follows at once from the definition that $\ell_j s_j\geqslant \lambda k_n$ and therefore, since a_k is non-increasing and $0<\rho<1$, we deduce from (27) that

$$a_\ell > c_2\rho^{\ell s_j} \quad \text{for} \quad \ell_j<\ell\leqslant \ell_{j+1}, \quad 1\leqslant j\leqslant m-1. \tag{28}$$

Next, apply (20) where n is replaced by k_n. Using the estimate (28) for $\ell > \ell_1$ and the estimate (26) for $1 \leqslant k \leqslant \ell_1$, we see from (24) on setting $c := \min(c_1, c_2)$ that

$$c^{k_n} \rho^{\alpha\ell_1(\ell_1+1)/2} \Pi_{j=1}^{m-1} \rho^{s_j\left(\ell_{j+1}(\ell_{j+1}+1)/2 - \ell_j(\ell_j+1)/2\right)} < C_\varepsilon^{k_n} (\rho+\varepsilon)^{k_n(k_n+1)/2}.$$

Raising both sides of this inequality to the power $2/k_n^2$ and letting n tend to infinity leads us to

$$\rho^{\lambda^2/\alpha} \Pi_{j=1}^{m-1} \rho^{(1/s_{j+1}^2 - 1/s_j^2)\lambda^2 s_j} < \rho + \varepsilon.$$

Since $0 < \rho < 1$ while $\varepsilon > 0$ was arbitrary, this amounts to saying that

$$\frac{\lambda^2}{\alpha} + \lambda^2 \sum_{j=1}^{m-1} s_j \left(\frac{1}{s_{j+1}^2} - \frac{1}{s_j^2} \right) \geqslant 1.$$

Rearranging the terms on the left-hand side and using that $s_j - s_{j+1} = \delta$ and $s_{j+1} \leqslant s_j$ yields *a fortiori* that

$$\frac{\lambda^2}{\alpha} + 2\lambda^2 \sum_{j=1}^{m-1} \frac{\delta}{s_{j+1}^2} \geqslant 1.$$

Further, by letting m tend to infinity, we get in the limit for this Riemann sum

$$\frac{\lambda^2}{\alpha} + 2\lambda^2 \int_\lambda^\alpha \frac{ds}{s^2} = 2\lambda - \frac{\lambda^2}{\alpha} \geqslant 1.$$

Therefore, as $\lambda > 1/2$, it holds that

$$\alpha \geqslant \frac{\lambda^2}{2\lambda - 1} > 0. \tag{29}$$

Now, if we had

$$\liminf_{k\to\infty} a_k^{1/k} > \rho^{\lambda^2/(2\lambda-1)},$$

then there would exist $0 < \alpha < \lambda^2/(2\lambda - 1)$ for which (25) would still hold, but this contradicts (29). Hence (23) is satisfied, at least with a less than or equal sign; the inequality, however, is strict because we could have taken λ slightly smaller in (22). □

We are now in a position to achieve the proofs of the two main results of this note.

4. Proof of Theorems 4 and 1

Proof of Theorem 4. First, *we claim* that it is enough to prove Theorem 4 in the special case where $A = \{|z| \geqslant 1\}$, so that $T = \mathbb{T}$ and $D = \mathbb{D}$. To see this, we use the notation introduced in the proof of Lemma 3. By conformal invariance of the condenser capacity, it holds that

$$\operatorname{cap}_D(K) = \operatorname{cap}_{\mathbb{D}}(\phi^{-1}(K)).$$

Let M_n^∞ be the best meromorphic approximant of f from $\mathcal{A}_n(D)$ while $M_n^\infty(f \circ \phi)$ and $M_n^\infty(F)$ denote the AAK approximants of $f \circ \phi$ and $F = \mathbb{P}_-(f \circ \phi)$, respectively. As explained in the proof of Lemma 3,

$$\|f - M_n^\infty\|_T = \|f \circ \phi - M_n^\infty(f \circ \phi)\|_{\mathbb{T}} = \|F - M_n^\infty(F)\|_{\mathbb{T}}. \tag{30}$$

Hence, if (9) holds for f, T, and K, then an analogous limiting inequality holds for F, $\mathbb{T}$ and $\phi^{-1}(K)$, whence the validity of the conclusion (10) for F, $\mathbb{T}$ and $\phi^{-1}(K)$ implies that it must also hold for f, T and K, *as claimed.*

Now, to prove Theorem 4 in the case of the unit disk, we use Lemma 5. Let L be a finite disjoint union of C^∞-smooth Jordan curves such that K lies in the interior of L, whose closure we continue to denote by K_L. Decompose Γ_f as in (15). Set

$$\begin{cases} \rho & = \exp\{-2/\mathrm{cap}_{\mathbb{D}}(K_L)\}, \\ a_k & = s_{k-1}(\Gamma_f), \\ b_k & = \|B_2\|\,\|B_3\|\,s_{k-1}(B_1)\,s_{k-1}(B_4). \end{cases}$$

With these choices we get from (18) and (16) that (20) and (21) are satisfied. Moreover, we deduce from (12) and (9) that (22) holds for any $\lambda > b/2$. The first conclusion of Lemma 5 states that $\lambda > 1/2$, which implies that $b \geqslant 1$. Moreover, if $b \in [1,2)$, then minimizing over $\lambda \in (b/2, 1)$ gives us

$$\begin{aligned} \liminf_{n\to\infty} \|f - M_n^\infty\|_T^{1/n} &\leqslant \inf_{b/2<\lambda<1} \exp\left\{-\frac{2\lambda^2}{(2\lambda-1)}\frac{1}{\mathrm{cap}_{\mathbb{D}}(K_L)}\right\} \\ &= \exp\left\{-\frac{b^2}{2(b-1)}\frac{1}{\mathrm{cap}_{\mathbb{D}}(K_L)}\right\}, \end{aligned}$$

by (23). As the left-hand side of the inequality above is independent of L and since $\mathrm{cap}_{\mathbb{D}}(K_L)$ can be made arbitrarily close to $\mathrm{cap}_{\mathbb{D}}(K)$, see (19), the conclusion (10) follows. □

Proof of Theorem 1. Given $\epsilon \in (0,1)$, let $T_\epsilon := \{\phi(z) : |z| = 1-\epsilon\}$ and D_ϵ be the interior domain of T_ϵ. Clearly, T_ϵ is a smooth Jordan curve for any continuum A. Fix ϵ small enough so that f extends analytically across T_ϵ. Let $M_{n,\epsilon}^\infty \in \mathcal{A}_n(D_\epsilon)$ be the best meromorphic approximant to f on T_ϵ. Write $M_{n,\epsilon} = r_n + g_n$, where $r_n \in \mathcal{R}_n(D_\epsilon)$ and $g_n \in \mathcal{A}(D_\epsilon)$. Then it follows from Cauchy's integral formula that

$$\begin{aligned} |(f-r_n)(z)| &= \left|\int_{T_\epsilon} \frac{(f-r_n)(s)}{z-s}\frac{ds}{2\pi\mathrm{i}}\right| = \left|\int_{T_\epsilon} \frac{(f-M_{n,\epsilon}^\infty)(s)}{z-s}\frac{ds}{2\pi\mathrm{i}}\right| \\ &\leqslant C_\epsilon \|f - M_{n,\epsilon}^\infty\|_{T_\epsilon}, \quad z \in T, \end{aligned}$$

for some constant C_ϵ. Put $\phi_\epsilon(z) := \phi(z(1-\epsilon))$, which maps $\mathbb{D}$ conformally onto D_ϵ. Set $F_\epsilon := \mathbb{P}_-(f \circ \phi_\epsilon)$. Then, the above estimate together with (30) and (12) yield that

$$\rho_n(f, A) \leqslant C_\epsilon s_n(\Gamma_{F_\epsilon}).$$

As in the previous proof, let L be a finite disjoint union of C^∞-smooth Jordan curves with $K \subset K_L \subset D_\epsilon$. Decompose Γ_{F_ϵ} as in (15) with respect to $\phi_\epsilon^{-1}(L)$. Then, similarly to the previous proof, we get from (16) and (18) along with the conformal equivalence of Greenian capacity that

$$\begin{cases} \prod_{k=1}^n \rho_{k-1}(f,A) \leqslant \prod_{k=1}^n b_k, \\ b_k = C_\epsilon \|B_2\|\,\|B_3\|\,s_{k-1}(B_1)\,s_{k-1}(B_4), \\ \lim_{k\to\infty} b_k^{1/k} = \exp\left\{-\frac{2}{\mathrm{cap}_{D_\epsilon}(K_L)}\right\}. \end{cases}$$

Since $\mathrm{cap}_{D_\epsilon}(K_L) = \mathrm{cap}_{\mathbb{C}\setminus K_L}(\mathbb{C}\setminus\overline{D}_\epsilon)$ by characteristic properties of Greenian equilibrium potentials [22, Theorems II.5.11 & II.5.12], we get from the right continuity of Greenian capacity on compact sets that this quantity can be made arbitrarily close to $\mathrm{cap}_D(K)$, see (19). Now, Theorem 1 follows from Lemma 5 exactly as Theorem 4 did. □

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