

ZEROS OF REAL RANDOM POLYNOMIALS SPANNED BY OPUC

MAXIM L. YATTSELEV AND AARON YEAGER

ABSTRACT. Let $\{\varphi_i\}_{i=0}^\infty$ be a sequence of orthonormal polynomials on the unit circle with respect to a probability measure μ . We study zero distribution of random linear combinations of the form

$$P_n(z) = \sum_{i=0}^{n-1} \eta_i \varphi_i(z),$$

where $\eta_0, \dots, \eta_{n-1}$ are i.i.d. standard Gaussian variables. We use the Christoffel-Darboux formula to simplify the density functions provided by Vanderbei for the expected number real and complex of zeros of P_n . From these expressions, under the assumption that μ is in the Nevai class, we deduce the limiting value of these density functions away from the unit circle. Under mere assumption that μ is doubling on subarcs of \mathbb{T} centered at 1 and -1 , we show that the expected number of real zeros of P_n is at most

$$(2/\pi) \log n + O(1),$$

and that the asymptotic equality holds when the corresponding recurrence coefficients decay no slower than $n^{-(3+\epsilon)/2}$, $\epsilon > 0$. We conclude with providing results that estimate the expected number of complex zeros of P_n in shrinking neighborhoods of compact subsets of \mathbb{T} .

1. INTRODUCTION AND MAIN RESULTS

In [3], Kac considered random polynomials

$$(1) \quad P_n(z) = \eta_0 + \eta_1 z + \dots + \eta_{n-1} z^{n-1},$$

where η_i are i.i.d. standard Gaussian variables. He has shown that $\mathbb{E}(N_n(\Omega))$, the expected number of zeros of P_n on a measurable set $\Omega \subset \mathbb{R}$, is equal to

$$(2) \quad \mathbb{E}(N_n(\Omega)) = \frac{1}{\pi} \int_{\Omega} \frac{\sqrt{1 - h_n^2(x)}}{|1 - x^2|} dx, \quad h_n(x) = \frac{nx^{n-1}(1 - x^2)}{1 - x^{2n}},$$

from which he proceeded with an estimate

$$(3) \quad \mathbb{E}(N_n(\mathbb{R})) = \frac{2 + o(1)}{\pi} \log n \quad \text{as } n \rightarrow \infty.$$

In fact, it was eventually shown by Wilkins [13] that $\mathbb{E}(N_n(\mathbb{R}))$ has an asymptotic expansion of the form

$$\mathbb{E}(N_n(\mathbb{R})) \sim \frac{2}{\pi} \log n + \sum_{k=0}^{\infty} A_k n^{-k}$$

for some constants A_k . In another connection, Shepp and Vanderbei [9] showed that

$$(4) \quad \mathbb{E}(N_n(\Omega)) = \int_{\Omega \cap \mathbb{R}} \rho_n^{(1,0)}(x) dx + \int_{\Omega} \rho_n^{(0,1)}(z) dx dy,$$

where $\Omega \subset \mathbb{C}$ is measurable, $\rho_n^{(1,0)}(x)$ is the integrand in (2), and $\rho_n^{(0,1)}(z)$ is the explicitly known intensity function for the complex zeros, see (6) and (7) further below.

The result of Kac was generalized in many directions, see the introduction of [11] for the references. In particular, one could replace the basis $\{z^i\}$ with an arbitrary set of entire functions $\{f_i(z)\}$ that are real on the real line. That is, instead of (1), one could consider random functions of the form

$$(5) \quad P_n(z) = \eta_0 f_0(z) + \eta_1 f_1(z) + \cdots + \eta_{n-1} f_{n-1}(z),$$

where η_i are again i.i.d. standard Gaussian variables. In this case Vanderbei [11] has shown that (4) still holds with

$$(6) \quad \rho_n^{(1,0)}(x) = \frac{1}{\pi} \frac{\sqrt{K_n(x,x)K_n^{(1,1)}(x,x) - K_n^{(1,0)}(x,x)^2}}{K_n(x,x)}$$

(in [5, Proposition 1.1], (6) was derived under mere assumption of continuous differentiability of f_i) and

$$(7) \quad \begin{aligned} \rho_n^{(0,1)}(z) &= \frac{1}{\pi} \frac{K_n^{(1,1)}(z,z)}{(K_n(z,z)^2 - |K_n(z,\bar{z})|^2)^{1/2}} \\ &- \frac{1}{\pi} \frac{K_n(z,z)(|K_n^{(1,0)}(z,z)|^2 + |K_n^{(1,0)}(z,\bar{z})|^2)}{(K_n(z,z)^2 - |K_n(z,\bar{z})|^2)^{3/2}} \\ &+ \frac{2 \operatorname{Re}(K_n(z,\bar{z})K_n^{(1,0)}(z,z)K_n^{(1,0)}(\bar{z},z))}{\pi (K_n(z,z)^2 - |K_n(z,\bar{z})|^2)^{3/2}}, \end{aligned}$$

where

$$(8) \quad \begin{cases} K_n(z,w) &:= \sum_{i=0}^{n-1} f_i(z)\overline{f_i(w)}, \\ K_n^{(1,0)}(z,w) &:= \sum_{i=0}^{n-1} f'_i(z)\overline{f_i(w)}, \\ K_n^{(1,1)}(z,w) &:= \sum_{i=0}^{n-1} f'_i(z)\overline{f'_i(w)}. \end{cases}$$

In this note we concentrate on the case where the functions $f_i = \varphi_i$ are real orthonormal polynomials on the unit circle complementing the case where $f_i = p_i$ are orthonormal polynomials on the real line [5]. That is, for some probability Borel measure μ on \mathbb{T} that is symmetric with respect to conjugation, it holds that

$$(9) \quad \int_{\mathbb{T}} \varphi_i(z)\overline{\varphi_j(z)}d\mu(z) = \delta_{ij},$$

where δ_{ij} is the usual Kronecker symbol. Notice that all the coefficients of the polynomials φ_i are real due to the conjugate-symmetry of the measure μ . Write $\varphi_i(z) = \kappa_i \Phi_i(z)$, where Φ_i is monic. Polynomials Φ_i satisfy Szegő recurrence relations

$$(10) \quad \begin{cases} \Phi_{i+1}(z) = z\Phi_i(z) - \alpha_i \Phi_i^*(z), \\ \Phi_{i+1}^*(z) = \Phi_i^*(z) - \alpha_i z\Phi_i(z), \end{cases}$$

where $\Phi_i^*(z) := z^i \Phi_i(1/z)$ and $\{\alpha_i\} \subset (-1, 1)$ are the recurrence coefficients, see [10, Theorem 1.5.2]. Recall also that given $\{\alpha_i\} \subset (-1, 1)$, there exists a unique conjugate-symmetric probability measure μ whose monic orthogonal polynomials satisfy (10), see [10, Theorem 1.7.11]. Moreover, in this case it holds that

$$(11) \quad \kappa_n = \prod_{i=0}^{n-1} (1 - \alpha_i^2)^{-1/2},$$

see [10, Equation (1.5.22)]. Thus, we might refer to orthonormal polynomials φ_i as defined by either the measure μ or the recurrence coefficients $\{\alpha_i\}$.

Theorem 1.1. *Let $\{\varphi_i\}$ be a sequence of polynomials satisfying (9). Further, let P_n be a real random polynomial (5) with $f_i = \varphi_i$ and η_i being i.i.d. standard Gaussian variables. Then the intensity function $\rho_n^{(1,0)}$ from (6) can be written as*

$$(12) \quad \rho_n^{(1,0)}(x) = \frac{1}{\pi} \frac{\sqrt{1-h_n^2(x)}}{|1-x^2|}, \quad h_n(x) = \frac{(1-x^2)b'_n(x)}{1-b_n^2(x)}, \quad b_n(z) = \frac{\varphi_n(z)}{\varphi_n^*(z)}.$$

Clearly, when the recurrence coefficients are all zero, $\varphi_n(z) = z^n$ and respectively $b_n(z) = z^n$. That is, we recover the intensity function from the Kac formula (2).

Proposition 1.2. *Under the conditions of Theorem 1.1, it holds that*

$$h_n(z) = \frac{1-z^2}{2} \frac{F'_n(z)}{F_n(z)},$$

where F_n is a trivial Caratheodory function given by

$$F_n(z) = -\frac{\Phi_n(z; -1)}{\Phi_n(z; 1)} = \int \frac{\zeta+z}{\zeta-z} d\sigma_n(\zeta),$$

σ_n is a probability measure defined by

$$\sigma_n = \sum_{k=1}^n \frac{|\varphi_{n-1}(\zeta_{k,n})|^2}{K_n(\zeta_{k,n}, \zeta_{k,n})} \delta_{\zeta_{k,n}},$$

δ_ζ is the unit point mass at ζ , $\zeta_{k,n} \in \mathbb{T}$ are the zeros of $\Phi_n(z; 1)$, and

$$\Phi_n(z; \beta) := z\Phi_{n-1}(z) - \beta\Phi_{n-1}^*(z), \quad |\beta| = 1,$$

is the n -th paraorthogonal polynomial¹ associated with parameter β .

Since the Blaschke products b_n necessarily satisfy $|b_n(z)| \leq 1$ in $\overline{\mathbb{D}}$, they form a normal family there. Moreover, as $b_n(1/z) = 1/b_n(z)$, we see that

$$(13) \quad \rho_n^{(1,0)}(1/x) = x^2 \rho_n^{(1,0)}(x).$$

The following corollary is immediate.

Corollary 1.3. *In the setting of Theorem 1.1, let $\mathcal{N} \subset \mathbb{N}$ be such that $b_n(z) \rightarrow b(z) \neq 1$ as $\mathcal{N} \ni n \rightarrow \infty$ for some analytic function $b(z)$ in \mathbb{D} . Then*

$$\rho_n^{(1,0)}(x) \rightarrow \frac{1}{\pi} \frac{\sqrt{1-h^2(x)}}{1-x^2}, \quad h(x) = b'(x) \frac{1-x^2}{1-b^2(x)},$$

locally uniformly on $(-1, 1)$ as $\mathcal{N} \ni n \rightarrow \infty$. In particular, if $\alpha_i \rightarrow 0$ as $i \rightarrow \infty$, then

$$\rho_n^{(1,0)}(x) \rightarrow \frac{1}{\pi} \frac{1}{|1-x^2|} \quad \text{as } n \rightarrow \infty$$

uniformly on closed subsets of $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ as $n \rightarrow \infty$.

The second claim of the corollary is a straightforward consequence of the fact that $b_n(z) \rightarrow 0$ locally uniformly in \mathbb{D} as $n \rightarrow \infty$ if and only if $\alpha_i \rightarrow 0$ as $i \rightarrow \infty$, see [10, Theorem 1.7.4].

Recall that a measure μ is called doubling on a subarc $T \subseteq \mathbb{T}$ if there exists a constant $L > 0$ such that

$$\mu(2I) \leq L\mu(I), \quad 2I \subseteq T,$$

for any subarc I , where $2I$ is a subarc of T with the same center as I and twice the arclength.

¹It is known that all the zeros of $\Phi_n(z; \beta)$ are simple and lie on the unit circle.

Theorem 1.4. *In the setting of Theorem 1.1, assume that there exist two subarcs of \mathbb{T} , centered at 1 and -1 , on which μ is doubling. Then it holds that*

$$(14) \quad \mathbb{E}(N_n(\mathbb{R})) \leq \frac{2}{\pi} \log n + \mathcal{O}(1).$$

Moreover, if the quantities $|k^p \alpha_k|$ are uniformly bounded above for some $p > 3/2$, then

$$(15) \quad \mathbb{E}(N_n(\mathbb{R})) = \frac{2 + o(1)}{\pi} \log n.$$

The assumption on α_k 's in (15) implies that they are absolutely summable. Hence, it follows from Baxter's theorem, see [10, Theorem 5.2.1], that μ is absolutely continuous with respect to the arclength distribution on \mathbb{T} and the Radon-Nikodym derivative is continuous and non-vanishing there. In particular, μ is doubling on \mathbb{T} .

Proposition 1.5. *In the setting of Theorem 1.1, assume that*

$$\mu = t\nu + (1-t)\delta_1, \quad t \in (0, 1),$$

where ν is a conjugate-symmetric probability measure on the unit circle such that $|k^p \alpha_k(\nu)|$ are uniformly bounded above for some $p > 3/2$. Then (15) holds while

$$(16) \quad \alpha_{n-1} = \alpha_{n-1}(\nu) + \varphi_{n-1}(1; \nu) \varphi_n(1; \nu) \frac{\sqrt{1 - |\alpha_{n-1}(\nu)|^2}}{t(1-t)^{-1} + K_n(1, 1; \nu)},$$

where the quantities $\alpha_n(\nu), \varphi_n(z; \nu), K_n(z, w; \nu)$ are defined as before only with respect to the measure ν .

Formula (16) was derived in [14] and shows that $\alpha_n \sim 1/n$ as $n \rightarrow \infty$, that is, the recurrence coefficients do not obey the conditions of Theorem 1.4. Indeed, the coefficients $\alpha_k(\nu)$ are absolutely summable. Thus, there exists a constant $c > 1$ such that $c^{-1} \leq |\varphi_k(1; \nu)| \leq c$ for all k , see [10, Equation (1.5.16)] and (11). Hence, $n/c^2 \leq K_n(1, 1; \nu) \leq nc^2$, which yields the claim.

Theorem 1.6. *In the setting of Theorem 1.1, assume that $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$. Then we have*

$$\rho_n^{(0,1)}(z) \rightarrow \frac{1}{\pi(1-|z|^2)^2} \sqrt{1 - \left| \frac{1-|z|^2}{1-z^2} \right|^2}$$

locally uniformly in $\overline{\mathbb{C}} \setminus (\mathbb{T} \cup \mathbb{R})$ as $n \rightarrow \infty$.

It follows from Theorem 1.6 that the zeros of P_n almost surely accumulate on the unit circle. In fact, the following discrepancy results hold.

Theorem 1.7. *In the setting of Theorem 1.1, assume that the measure of orthogonality μ in (9) is regular in the sense of Ullman-Stahl-Totik, that is,*

$$(17) \quad \varepsilon_n := \frac{1}{n} \log |\kappa_n| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where κ_n is the leading coefficient of φ_n . Then for any subarc $S \subset \mathbb{T}$, it holds that

$$(18) \quad \mathbb{E} \left[\left| \frac{1}{n} N(\Omega(S, \delta)) - \frac{|S|}{2\pi} \right| \right] \leq \frac{C}{\delta} \sqrt{\frac{\log n}{n} + \sqrt{\varepsilon_n}} \quad \text{as } n \rightarrow \infty,$$

where C is independent of n and $\Omega(S, \delta) := \{rz : z \in S, r \in (1-\delta, 1+\delta)\}$.

The claim of Theorem 1.7 is contained in [8, Corollary 3.2] under the additional conditions that the measure μ for the basis $\{\varphi_i\}$ is absolutely continuous with Radon-Nikodym derivative bounded below by a positive constant on the whole unit circle. We show that the proof of [8, Corollary 3.2] in fact remains valid under restriction (17).

It follows from (18) that the zeros of P_n that are expected to approach an arc S are contained in an annular neighborhood of width $(\frac{\log n}{n} + \sqrt{\epsilon_n})^{1/2-\epsilon}$ for any $\epsilon > 0$. More stringent assumptions on the measure μ allow us to decrease the width this neighborhood.

Theorem 1.8. *In the setting of Theorem 1.1, assume that $\alpha_i \rightarrow 0$ as $i \rightarrow \infty$. Let S be a compact subset of $\mathbb{T} \setminus \{\pm 1\}$. Assume, in addition, that μ is absolutely continuous with respect to the arclength measure on an open set containing S and its Radon-Nikodym derivative is positive and continuous at each point of S . Given $-\infty < \tau_1 < \tau_2 < \infty$, it follows that*

$$(19) \quad \frac{1}{n} \mathbb{E}[N(\Omega(S, \tau_1, \tau_2))] \rightarrow \frac{|S|}{2\pi} \left(\frac{H'(\tau_2)}{H(\tau_2)} - \frac{H'(\tau_1)}{H(\tau_1)} \right) \quad \text{as } n \rightarrow \infty,$$

where $\Omega(S, \tau_1, \tau_2) := \{rz : z \in S, r \in (1 + \frac{\tau_1}{2n}, 1 + \frac{\tau_2}{2n})\}$ and $H(\tau) := \frac{e^\tau - 1}{\tau}$.

It can be readily verified that H'/H is increasing on the real line with

$$\lim_{\tau \rightarrow -\infty} \frac{H'(\tau)}{H(\tau)} = 0 \quad \text{and} \quad \frac{H'(\tau)}{H(\tau)} = 1 - \frac{H'(-\tau)}{H(-\tau)}.$$

Hence, the zeros of P_n approaching S are expected to be contained in an annular band around S of width $n^{-1+\epsilon}$ for any $\epsilon > 0$.

2. PROOF OF THEOREM 1.1 AND PROPOSITION 1.2

2.1. Proof of Theorem 1.1. According to the Christoffel-Darboux formula [10, Theorem 2.2.7] and since polynomials φ_n have real coefficients, it holds that

$$(20) \quad K_n(z, w) = \frac{\varphi_n^*(z)\varphi_n^*(\bar{w}) - \varphi_n(z)\varphi_n(\bar{w})}{1 - z\bar{w}}.$$

Hence,

$$(21) \quad K_n^{(1,0)}(z, w) = \frac{(\varphi_n^*)'(z)\varphi_n^*(\bar{w}) - \varphi_n'(z)\varphi_n(\bar{w})}{1 - z\bar{w}} + \bar{w} \frac{K_n(z, w)}{1 - z\bar{w}}$$

and

$$(22) \quad \begin{aligned} K_n^{(1,1)}(z, w) &= \frac{(\varphi_n^*)'(z)(\varphi_n^*)'(\bar{w}) - \varphi_n'(z)\varphi_n'(\bar{w})}{1 - z\bar{w}} + z \frac{(\varphi_n^*)'(z)\varphi_n^*(\bar{w}) - \varphi_n'(z)\varphi_n(\bar{w})}{(1 - z\bar{w})^2} \\ &\quad + \bar{w} \frac{\varphi_n^*(z)(\varphi_n^*)'(\bar{w}) - \varphi_n(z)\varphi_n'(\bar{w})}{(1 - z\bar{w})^2} + \frac{(1 + z\bar{w})K_n(z, w)}{(1 - z\bar{w})^2}. \end{aligned}$$

Thus,

$$K_n(x, x)K_n^{(1,1)}(x, x) - K_n^{(1,0)}(x, x)^2 = \frac{K_n^2(x, x)}{(1 - x^2)^2} - \left(\frac{\varphi_n^*(x)\varphi_n'(x) - \varphi_n(x)(\varphi_n^*)'(x)}{1 - x^2} \right)^2.$$

Therefore, the claim of the theorem now follows from (6) since

$$\rho_n^{(1,0)}(x)^2 = \frac{1}{\pi^2} \left[\frac{1}{(1 - x^2)^2} - \left(\frac{b_n'(x)}{1 - b_n^2(x)} \right)^2 \right] = \frac{1}{\pi^2} \frac{1 - h_n^2(x)}{(1 - x^2)^2}.$$

2.2. Proof of Proposition 1.2. It can be readily verified using recurrence relations (10) that

$$(23) \quad h_n(z) = (1 - z^2) \frac{(zb_{n-1}(z))'}{1 - (zb_{n-1}(z))^2}.$$

Therefore, the first claim of the proposition holds with

$$F_n(z) := \frac{1 + zb_{n-1}(z)}{1 - zb_{n-1}(z)}.$$

Since $zb_{n-1}(z)$ is a finite Blaschke product, $F_n(z)$ is a trivial Caratheodory function [10, Section 1.3]. Clearly,

$$F_n(z) = -\frac{z\Phi_{n-1}(z) + \Phi_{n-1}^*(z)}{z\Phi_{n-1}(z) - \Phi_{n-1}^*(z)} = -\frac{\Phi_n(z; -1)}{\Phi_n(z; 1)}$$

by the very definition of $b_{n-1}(z)$ and $\Phi_n(z; \beta)$. Thus, it only remains to prove the integral representation. On the one hand, it holds that

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\sigma_n = -\sum_{k=1}^n \frac{|\varphi_{n-1}(\zeta_{k,n})|^2}{K_n(\zeta_{k,n}, \zeta_{k,n})} \frac{z + \zeta_{k,n}}{z - \zeta_{k,n}} = -\left(\sum_{k=1}^n \frac{|\varphi_{n-1}(\zeta_{k,n})|^2}{K_n(\zeta_{k,n}, \zeta_{k,n})} \right) \frac{z^n + \cdots}{z^n + \cdots},$$

which is a rational function of type (n, n) with a simple pole at each $\zeta_{k,n}$ with residue

$$-2\zeta_{k,n} \frac{|\varphi_{n-1}(\zeta_{k,n})|^2}{K_n(\zeta_{k,n}, \zeta_{k,n})}.$$

On the other hand, by the very definition, $F_n(z)$ is also a rational function of type (n, n) with a simple pole at each $\zeta_{k,n}$. Next observe that

$$(24) \quad \Phi_n(z; 1) = \frac{z - \zeta_{k,n}}{\kappa_{n-1}\varphi_{n-1}(\zeta_{n,k})} K_n(z, \zeta_{k,n}),$$

where κ_{n-1} is the leading coefficient of $\varphi_{n-1}(z)$. Indeed, the normalization follows from the fact that the left-hand side is a monic polynomial. Furthermore, Christoffel-Darboux formula (20) gives

$$(25) \quad \begin{aligned} \kappa_n^{-2} \bar{\zeta}(\zeta - z) K_n(z, \zeta) &= \Phi_n^*(z) \overline{\Phi_n^*(\zeta)} - \Phi_n(z) \overline{\Phi_n(\zeta)} \\ &= \Phi_{n-1}^*(z) (\overline{\Phi_n^*(\zeta)} + \alpha_{n-1} \overline{\Phi_n(\zeta)}) - z \Phi_{n-1}(z) (\overline{\alpha_{n-1} \Phi_n^*(\zeta)} + \overline{\Phi_n(\zeta)}) \end{aligned}$$

for any $\zeta \in \mathbb{T}$, where we used (10) for the second equality. Now, since

$$(26) \quad \zeta_{k,n} \Phi_{n-1}(\zeta_{k,n}) = \Phi_{n-1}^*(\zeta_{k,n}),$$

we get from (10) that

$$\Phi_n^*(\zeta_{k,n}) + \alpha_{n-1} \Phi_n(\zeta_{k,n}) = \Phi_{n-1}^*(\zeta_{k,n})(1 - \alpha_{n-1}^2) = \alpha_{n-1} \Phi_n^*(\zeta_{k,n}) + \Phi_n(\zeta_{k,n}).$$

Combining the above equality with (25) finishes the justification of (24). Now, (24) and (26) yield that the residue of F_n at $\zeta_{k,n}$ is equal to

$$-2\zeta_{k,n} \Phi_{n-1}(\zeta_{k,n}) \frac{\kappa_{n-1} \overline{\varphi_{n-1}(\zeta_{n,k})}}{K_n(\zeta_{k,n}, \zeta_{k,n})} = -2\zeta_{k,n} \frac{|\varphi_{n-1}(\zeta_{n,k})|^2}{K_n(\zeta_{k,n}, \zeta_{k,n})},$$

which proves the integral representation. In particular, since $F_n(z) = -1 + O(1/z)$ as $z \rightarrow \infty$, it must hold that

$$\sum_{k=1}^n \frac{|\varphi_{n-1}(\zeta_{k,n})|^2}{K_n(\zeta_{k,n}, \zeta_{k,n})} = 1,$$

that is, σ_n is a probability measure.

3. PROOF OF THEOREM 1.4 AND PROPOSITION 1.5

In what follows, to avoid complicated schemes of labeling constants, we shall write

$$f_n(z) \lesssim g_n(z), \quad z \in K, n \in \mathbb{N} \quad \Leftrightarrow \quad f_n(z) \leq Cg_n(z), \quad z \in K, n \in \mathbb{N},$$

where the constant C depends possibly on K but not on z . Furthermore, we write

$$f_n(z) \sim g_n(z) \quad \Leftrightarrow \quad f_n(z) \lesssim g_n(z) \lesssim f_n(z).$$

3.1. An Auxiliary Estimate. Recall that the n -th Christoffel function of μ is given by

$$(27) \quad \lambda_n(z; \mu) := \inf_{\deg(p) \leq n-1} |p(z)|^{-2} \int |p|^2 d\mu = K_n^{-1}(z, z),$$

where the last equality is extremely well known, see for example [10, Equation (1.2.39)].

In this subsection we prove the following claim: *if the measure μ is doubling on a subarc $T \subset \mathbb{T}$, then it holds that*

$$(28) \quad \lambda_n(ze^{ia/n}; \mu) \sim \mu_n(z) := \int_{T(z, \frac{1}{n})} d\mu, \quad z \in T', \quad |a| \leq 2,$$

uniformly with respect to z, a, n , where $T' \subset T$ is a subarc with endpoints different from those of T and $T(z, \delta)$ stands for the subarc of \mathbb{T} centered at z of arclength 2δ . When μ is doubling on the whole circle \mathbb{T} and $a = 0$, this claim is simply [6, Theorem 4.3]. The proof of the localized version (28) is quite similar to the one of [6, Theorem 4.3]. However, to improve readability, we adduce the full proof of this fact below.

Given an integer m that we shall fix later, put

$$\begin{aligned} S_n(z, \eta) &:= \gamma_n \left(\sum_{k=0}^{n-1} \left(\frac{z}{\eta} \right)^k \right)^m \left(\sum_{k=0}^{n-1} \left(\frac{\eta}{z} \right)^k \right)^m \\ &= \gamma_n \left(\frac{1}{z\eta} \right)^{m(n-1)} \left(\frac{z^n - \eta^n}{z - \eta} \right)^{2m} = \gamma_n \left(\frac{\sin(\frac{a-b}{2})}{\sin(\frac{a-b}{2n})} \right)^{2m} \end{aligned}$$

where $z = e^{ia/n}$, $\eta = e^{ib/n}$, and the normalizing constant γ_n is chosen so that $\int_{\mathbb{T}} S_n(z, \eta) |d\eta| = 1$. It is known that $\gamma_n \sim n^{-2m+1}$. The last representation of $S_n(z, \eta)$ shows that

$$(29) \quad \operatorname{Re} \left(S_n(e^{ia/n}, e^{ib/n}) \right) \gtrsim n.$$

locally uniformly for $|a - b| < 2\pi$. Similarly, we can easily see from the second representation that

$$(30) \quad |S_n(z, \eta)| \lesssim \begin{cases} n, & |z - \eta| \leq \frac{1}{n}, \\ n^{-2m+1} |z - \eta|^{-2m}, & |z - \eta| \geq \frac{1}{n}, \end{cases}$$

for $|n(|z| - 1)|, |n(|\eta| - 1)| \leq A$, where the constant is uniform in $A > 0$.

We start with an upper bound. Let $z \in T'$ and $|a| \leq 2$. Since $S_n(ze^{ia}, ze^{ia}) = \gamma_n n^{2m} \sim n$, it is immediate from (27) that

$$(31) \quad \begin{aligned} \lambda_n(ze^{ia}; \mu) &\leq |S_{\lfloor n/2m \rfloor}(ze^{ia}, ze^{ia})|^{-2} \int_{\mathbb{T}} |S_{\lfloor n/2m \rfloor}(ze^{ia}, \eta)|^2 d\mu(\eta) \\ &\lesssim \frac{1}{n} \int_T |S_{\lfloor n/2m \rfloor}(ze^{ia}, \eta)|^2 \mu_n(\eta) |d\eta| + \frac{1}{n^2} \int_{\mathbb{T} \setminus T} |S_{\lfloor n/2m \rfloor}(ze^{ia}, \eta)|^2 d\mu(\eta), \end{aligned}$$

where the inequality on T follows from [1, Equation (4.25)]. It is known, see for example [6, Lemma 2.1(ix)], that doubling property is equivalent to

$$(32) \quad \mu_n(\eta) \lesssim (1 + n|z - \eta|)^s \mu_n(z),$$

uniformly $z, \eta \in T$, where the parameter s depends only on the constant L in the doubling inequality $\mu(2I) \leq L\mu(I)$. Choose $m > (s + 1)/2$. Then (30) and (32) yield that the first integral in (31) can be estimates above by a constant times

$$(33) \quad \frac{\mu_n(z)}{n} \int_T |S_{\lfloor n/2m \rfloor}(ze^{ia}, \eta)|^2 (1 + n|z - \eta|)^s |d\eta| \\ \lesssim n\mu_n(z) \int_{T(z, \frac{1}{n})} |d\eta| + \frac{\mu_n(z)}{n^{4m-s-1}} \int_{T \setminus T(z, \frac{1}{n})} \frac{|d\eta|}{|z - \eta|^{4m-s}} \lesssim \mu_n(z).$$

To estimate the second integral in (31), let us point out that (32) is a consequence of the inequality

$$\int_I d\mu \lesssim \left(\frac{|I| + |J| + \text{dist}(I, J)}{|J|} \right)^s \int_J d\mu, \quad I, J \subseteq T,$$

where the constant is independent of I, J , see [6, Lemma 2.1(viii)]. Therefore,

$$(34) \quad \mu_n(z) \gtrsim n^{-s}, \quad z \in T.$$

Thus, our choice of m , (30), and (34) imply that

$$(35) \quad \frac{1}{n^2} \int_{\mathbb{T} \setminus T} |S_{\lfloor n/2m \rfloor}(z, \eta)|^2 d\mu(\eta) \lesssim \frac{1}{n^{4m}} \int_{\mathbb{T} \setminus T} \frac{d\mu(\eta)}{|z - \eta|^{4m}} \lesssim \frac{1}{n^{4m}} \lesssim \frac{1}{n^{2s+2}} \lesssim \mu_n(z).$$

The upper bound in (28) follows now by plugging estimates (33) and (35) into (31).

It only remains to prove the lower bound in (28). Let $z \in T'$ and $|a| \leq 2$. Define

$$Q_n(w) := w^{m(\lfloor n/2m \rfloor - 1)} \int_{\mathbb{T}} S_{\lfloor n/2m \rfloor}(w, \eta) (n\mu_n(\eta))^{1/2} |d\eta|,$$

which is a polynomial of degree at most $n - 2m$. We get from (29) and (32) that

$$(36) \quad |Q_n(ze^{ia/n})| \gtrsim \left| \int_{T(z, \frac{1}{n})} \text{Re} \left(S_{\lfloor n/2m \rfloor}(ze^{ia/n}, \eta) \right) (n\mu_n(\eta))^{1/2} |d\eta| \right| \\ = \left| \int_{-1}^1 \frac{1}{n} \text{Re} \left(S_{\lfloor n/2m \rfloor}(e^{ia/n}, e^{it/n}) \right) (n\mu_n(ze^{it/n}))^{1/2} dt \right| \\ \gtrsim \int_{-1}^1 (n\mu_n(ze^{it/n}))^{1/2} dt \gtrsim (n\mu_n(z))^{1/2}.$$

As $S_n(w, \eta)$ is positive for $w, \eta \in \mathbb{T}$, it follows from the normalization of S_n and Jensen's inequality that

$$|Q_n(z)|^2 \leq \left(\int_T + \int_{\mathbb{T} \setminus T} \right) S_{\lfloor n/2m \rfloor}(z, \eta) n\mu_n(\eta) |d\eta|.$$

Similarly to (33), the first integral above can be estimated as follows:

$$\int_T S_{\lfloor n/2m \rfloor}(z, \eta) n\mu_n(\eta) |d\eta| \lesssim n\mu_n(z) \int_T (1 + n|z - \eta|)^s S_{\lfloor n/2m \rfloor}(z, \eta) |d\eta| \lesssim n\mu_n(z),$$

where the first estimate follows from (32) and the second one from (30). Moreover, we also have that

$$\int_{\mathbb{T} \setminus T} S_{\lfloor n/2m \rfloor}(z, \eta) n\mu_n(\eta) |d\eta| \lesssim n \int_{\mathbb{T} \setminus T} S_{\lfloor n/2m \rfloor}(z, \eta) |d\eta| \lesssim \frac{1}{n^{2(m-1)}} \leq \frac{n}{n^s} \lesssim n\mu_n(z),$$

where we again used (30) as well as (34). Altogether, we get that

$$(37) \quad |Q_n(z)|^2 \lesssim n\mu_n(z), \quad z \in T'.$$

Let T_n be a polynomial of degree at most $n - 1$ normalized to have value 1 at $ze^{ia/n}$. It follows from (36) and (37) that

$$\begin{aligned} \int |T_n|^2 d\mu &\geq \int_{T'} |T_n|^2 d\mu \gtrsim \int_{T'} |T_n(\eta)|^2 n\mu_n(\eta) |d\eta| \gtrsim \int_{T'} |(T_n Q_n)(\eta)|^2 |d\eta| \\ &= |Q_n(ze^{ia/n})|^2 \int_{T'} \frac{|(T_n Q_n)(\eta)|^2}{|Q_n(ze^{ia/n})|^2} |d\eta| \gtrsim n\mu_n(z) \int_{T'} \frac{|(T_n Q_n)(\eta)|^2}{|Q_n(ze^{ia/n})|^2} |d\eta|, \end{aligned}$$

where we also used [1, Equation (4.25)] for the second inequality. Since the polynomial in the last integral above is normalized to have value 1 at $ze^{ia/n}$ and is of degree at most $2n$, (27) yields that

$$(38) \quad \lambda_n(ze^{ia/n}; \mu) \gtrsim n\mu_n(z) \lambda_{2n}(ze^{ia/n}; \sigma),$$

where σ is the arclength measure on T' . Now, we know from [12, Theorem 2.4] that

$$(39) \quad n\lambda_{2n}(z; \sigma) = nK_{2n}^{-1}(z, z; \sigma) \gtrsim 1, \quad z \in T',$$

with the constant independent of z and n , where $K_{2n}(z, w; \sigma)$ is defined exactly as in (8) with $f_i(z) = \varphi_i(z; \sigma)$ and $\varphi_i(z; \sigma)$ being i -th orthonormal polynomial with respect to σ . Then since

$$|K_{2n}(z, w; \sigma)| \leq \sqrt{K_{2n}(z, z; \sigma)} \sqrt{K_{2n}(w, w; \sigma)},$$

which is simply an application of the Cauchy-Schwarz inequality, and by applying Bernstein-Walsh inequality to each variable of $K_{2n}(z, w; \sigma)$ as it was done in [4, Lemma 6.2], we get from (39) that

$$|K_{2n}(ue^{ib/n}, ve^{ic/n}; \sigma)| \lesssim n, \quad u, v \in T', \quad |b|, |c| \leq 2.$$

The lower bound in (28) now follows by restricting the above inequality to the diagonal and plugging it in (38).

3.2. Upper Estimate. It follows from (4), (12), and (13) that

$$\mathbb{E}(N_n) = 2 \int_{-1}^1 \rho_n^{(1,0)}(x) dx \leq \frac{2}{\pi} \log n + \mathcal{O}(1) + 2 \left(\int_{-1}^{-1+1/n} + \int_{1-1/n}^1 \right) \rho_n^{(1,0)}(x) dx.$$

We would like to show that the last two integrals are bounded above by an absolute constant. We shall show this only for the integral on $[1 - \frac{1}{n}, 1]$, the case of the other one being completely identical. It holds that

$$(40) \quad \int_{1-1/n}^1 \rho_n^{(1,0)}(x) dx \leq \int_{1-1/n}^1 \sqrt{\frac{K_n^{(1,1)}(x, x)}{K_n(x, x)}} \frac{dx}{\pi} = \int_0^1 \sqrt{\frac{K_n^{(1,1)}(1 - \frac{y}{n}, 1 - \frac{y}{n})}{n^2 K_n(1 - \frac{y}{n}, 1 - \frac{y}{n})}} \frac{dy}{\pi}.$$

which is an easy consequence of (6). As μ is doubling in some neighborhood of 1, it follows from the Cauchy-Schwarz inequality, (27), and the lower bound in (28) that

$$\left| K_n \left(1 + \frac{u}{n}, 1 + \frac{\bar{v}}{n} \right) \right| \leq K_n^{1/2} \left(1 + \frac{u}{n}, 1 + \frac{u}{n} \right) K_n^{1/2} \left(1 + \frac{\bar{v}}{n}, 1 + \frac{\bar{v}}{n} \right) \lesssim \mu_n^{-1}(1)$$

for $|u|, |v| \leq 3/2$. Consequently, the Cauchy integral formula for derivatives of holomorphic functions gives us

$$\left| K_n^{(1,1)} \left(1 + \frac{u}{n}, 1 + \frac{\bar{v}}{n} \right) \right| = \left| \frac{1}{(2\pi i)^2} \int_{|\eta|=3/2} \int_{|\xi|=3/2} \frac{K_n(1 + \frac{\eta}{n}, 1 + \frac{\bar{\xi}}{n})}{(\frac{\eta}{n} - \frac{u}{n})^2 (\frac{\bar{\xi}}{n} - \frac{\bar{v}}{n})^2} \frac{d\eta}{n} \frac{d\bar{\xi}}{n} \right| \lesssim \frac{n^2}{\mu_n(1)}$$

for $|u|, |v| \leq 1$. The desired claim now follows from the above inequality combined with the upper estimate in (28).

3.3. Lower Estimate. Under the current assumptions the measure μ is doubling on the whole circle (see the explanation after the statement of the theorem) and therefore we only need to prove the lower estimate. To this end, observe that

$$\mathbb{E}(N_n) > \frac{2}{\pi} \int_{-1+\frac{\log n}{n}}^{1-\frac{\log n}{n}} \frac{\sqrt{1-h_n^2(x)}}{1-x^2} dx \geq -\frac{2}{\pi} \sqrt{1-M_n^2} \log \left(\frac{\log n}{n} \right),$$

where M_n is the maximum of $|h_n(x)|$ on the interval of integration above. Thus, to prove (15) it is enough to show that $M_n = o(1)$ as $n \rightarrow \infty$.

By the conditions of the theorem the sequence of the recurrence coefficients is absolutely summable. Hence, it follows from [10, Theorem 1.5.3] that

$$(41) \quad 1 \lesssim |\Phi_n^*| \lesssim 1$$

uniformly on $\overline{\mathbb{D}}$. As $|\Phi_n| = |\Phi_n^*|$ on \mathbb{T} , it also follows from the Bernstein-Walsh inequality that

$$(42) \quad |\Phi_n(z)| \lesssim |z|^n, \quad |z| \geq 1.$$

It can be readily checked using (10) that

$$\Phi_n^*(z) = 1 - z \sum_{k=0}^{n-1} \alpha_k \Phi_k(z).$$

Therefore, we get from (42) and the absolute summability of α_k 's that

$$(43) \quad |\Phi_n^*(z)| \lesssim \sum_{k=0}^{n-1} |\alpha_k| |z|^{k+1} = \left(\sum_{k=0}^{m-1} + \sum_{k=m}^{n-1} \right) |\alpha_k| |z|^{k+1} \lesssim |z|^m + \Lambda_m |z|^n$$

for $|z| \geq 1$, where $\Lambda_m := \sum_{k=m}^{\infty} |\alpha_k|$. That is, it holds that

$$(44) \quad |\Phi_n(z)| \lesssim \Lambda_m + |z|^{n-m}, \quad |z| \leq 1.$$

Combining the above inequality with the lower bound in (41), we see that

$$(45) \quad |b_n(z)| \lesssim \Lambda_m + |z|^{n-m}, \quad |z| \leq 1.$$

It further follows from the Cauchy integral formula for the derivatives that

$$(46) \quad |(zb_{n-1}(z))'| \leq \int_{|\zeta|=r} \frac{|\zeta b_{n-1}(\zeta)|}{|\zeta - z|^2} \frac{|d\zeta|}{2\pi} \leq r^2 \frac{\Lambda_m + r^{n-m-1}}{r^2 - |z|^2}, \quad |z| < r.$$

On the other hand, Bernstein inequality for polynomials on the disk of radius r , (41), and (44) yield that

$$(47) \quad \max_{|z| \leq r} |(zb_{n-1}(z))'| \lesssim n (\Lambda_m + r^{n-m-1}).$$

Now, take m to be the integer part of $n/\log n$ and recall that $\Lambda_m \lesssim m^{1-p}$ according to the condition placed on the recurrence coefficients. Thus, inequalities (46) and (47), both applied with $r = 1 - \log n/n$, give

$$(48) \quad |(zb_n(z))'| \lesssim (\log n)^{p-1} \begin{cases} n^{3/2-p}, & |z| \leq 1 - n^{-1/2}, \\ n^{2-p}, & 1 - n^{-1/2} < |z| \leq 1 - \frac{\log n}{n}. \end{cases}$$

It also follows from (45) that for $x \in (-1, 1)$ we have that

$$\frac{1-x^2}{1-(xb_{n-1}(x))^2} \lesssim \begin{cases} 1, & |x| \leq 1 - n^{-1/2}, \\ n^{-1/2}, & 1 - n^{-1/2} < |x| \leq 1 - \frac{\log n}{n}. \end{cases}$$

Therefore, we deduce from (23) that $M_n \lesssim (\log n)^{p-1} n^{3/2-p} = o(1)$ as desired.

3.4. Proof of Proposition 1.5. Notice that the upper bound (14) remains valid in this case. We prove the lower bound as in the previous section by showing that the maximum of $|h_{n+1}(x)|$ on $[-1 + \log n/n, 1 - \log n/n]$ behaves like $o(1)$ as $n \rightarrow \infty$.

It was discovered by Geronimus [2], see also [14], that

$$\begin{aligned}\Phi_n(z) &= \Phi_n(z; \nu) - \frac{\Phi_n(1; \nu)K_n(z, 1; \nu)}{t(1-t)^{-1} + K_n(1, 1; \nu)} \\ &= \Phi_n(z; \nu) - \beta_n \frac{\Phi_n^*(z; \nu) - \Phi_n(z; \nu)}{1-z},\end{aligned}$$

where we used Christoffel-Darboux formula (20) and set

$$\beta_n := \frac{\varphi_n^2(1; \nu)}{t(1-t)^{-1} + K_n(1, 1; \nu)}.$$

Put $s_n(z) := zb_n(z; \nu)$. Then it holds that

$$zb_n(z) = \frac{(1-z)s_n(z) - \beta_n(z - s_n(z))}{1 - z(1 + \beta_n) + \beta_n s_n(z)} = 1 - \frac{(1-z)(1 - s_n(z))}{1 - z(1 + \beta_n) + \beta_n s_n(z)}.$$

Therefore,

$$1 - (zb_n(z))^2 = (1-z)^2(1 - s_n(z)) \frac{1 + s_n(z) + 2(s_n(z) - z) \frac{\beta_n}{1-z}}{(1 - z(1 + \beta_n) + \beta_n s_n(z))^2}$$

and

$$(zb_n(z))' = \frac{(1-z)^2(1 + \beta_n)s_n'(z) + \beta_n(2s_n(z) - 1)}{(1 - z(1 + \beta_n) + \beta_n s_n(z))^2}.$$

Thus, we get from (23) that

$$(49) \quad h_{n+1}(z) = \frac{(1-z^2)(1 + \beta_n)s_n'(z) + \beta_n \frac{1+z}{1-z}(2s_n(z) - 1)}{(1 - s_n(z))(1 + s_n(z) + \frac{2\beta_n}{1-z}(s_n(z) - z))}.$$

It follows from the explanation given after the statement of the proposition that $\beta_n \sim 1/n$ and therefore

$$\beta_n \frac{1+x}{1-x} |2s_n(x) - 1| \lesssim \frac{\beta_n}{1-x} \lesssim \frac{1}{\log n}, \quad -1 \leq x \leq 1 - \frac{\log n}{n},$$

where we used the fact that $|s_n(z)| \leq 1$ in \mathbb{D} . It also follows from (48) that

$$(1-x^2)(1 + \beta_n)|s_n'(x)| \lesssim (\log n)^{p-1} n^{3/2-p}, \quad |x| \leq 1 - \frac{\log n}{n}.$$

Hence, the numerator of (49) is of order $o(1)$ as $n \rightarrow \infty$ on the interval of interest. Similarly, we see from (45) that the denominator behaves like $1 + o(1)$ there, which finishes the proof of the proposition.

4. PROOF OF THEOREM 1.6

Let us modify expression (7) for $\rho_n^{(0,1)}$ to make it more amenable to the asymptotic analysis. Write

$$(50) \quad \pi (K_n(z, z)^2 - |K_n(z, \bar{z})|^2)^{3/2} \rho_n^{(0,1)}(z) := S_1(z) + S_2(z) + S_3(z),$$

where the functions $S_i(z)$ correspond to the respective summands in (7). For brevity, put

$$S_n(z, w) := (\varphi_n^*)'(z)\varphi_n^*(\bar{w}) - \varphi_n'(z)\varphi_n(\bar{w}).$$

Then it follows from (20)–(22) that $S_1(z)$ is equal to

$$(51) \quad \frac{1 + |z|^2}{(1 - |z|^2)^2} K_n(z, z) (K_n(z, z)^2 - |K_n(z, \bar{z})|^2)$$

$$(52) \quad + 2 \frac{\operatorname{Re}(z S_n(z, z))}{(1 - |z|^2)^2} (K_n(z, z)^2 - |K_n(z, \bar{z})|^2)$$

$$(53) \quad + \frac{|(\varphi_n^*)'(z)|^2 - |\varphi_n'(z)|^2}{1 - |z|^2} (K_n(z, z)^2 - |K_n(z, \bar{z})|^2),$$

$S_2(z)$ is equal to

$$(54) \quad -|z|^2 K_n(z, z) \left(\frac{K_n(z, z)^2}{(1 - |z|^2)^2} - \frac{|K_n(z, \bar{z})|^2}{|1 - z^2|^2} \right)$$

$$(55) \quad - 2 \frac{K_n(z, z)^2 \operatorname{Re}(z S_n(z, z))}{(1 - |z|^2)^2} - 2 \frac{K_n(z, z) \operatorname{Re}(z K_n(z, \bar{z}) \overline{S_n(z, \bar{z})})}{|1 - z^2|^2}$$

$$(56) \quad - \frac{K_n(z, z) |S_n(z, z)|^2}{(1 - |z|^2)^2} - \frac{K_n(z, z) |S_n(z, \bar{z})|^2}{|1 - z^2|^2},$$

and $S_3(z)$ is equal to

$$(57) \quad \frac{K_n(z, z) |K_n(z, \bar{z})|^2}{1 - |z|^2} \left(\frac{1 - |z|^4}{|1 - z^2|^2} - 1 \right)$$

$$(58) \quad + 2 \frac{|K_n(z, \bar{z})|^2}{1 - |z|^2} \operatorname{Re} \left(\frac{\bar{z} S_n(z, z)}{1 - \bar{z}^2} \right) + 2 \frac{K_n(z, z)}{1 - |z|^2} \operatorname{Re} \left(\frac{\bar{z} K_n(z, \bar{z}) \overline{S_n(z, \bar{z})}}{1 - \bar{z}^2} \right)$$

$$(59) \quad + \frac{2}{1 - |z|^2} \operatorname{Re} \left(\frac{K_n(z, \bar{z}) S_n(z, z) \overline{S_n(z, \bar{z})}}{1 - \bar{z}^2} \right),$$

where we used the identity $2\operatorname{Re}(z^2) = 1 + |z|^4 - |1 - z^2|^2$ in (57). Then we can rewrite (50) as

$$(60) \quad \pi (K_n(z, z)^2 - |K_n(z, \bar{z})|^2)^{3/2} \rho_n^{(0,1)}(z) := \Sigma_{n,1}(z) - \Sigma_{n,2}(z) + \Sigma_{n,3}(z),$$

where $\Sigma_{n,1}$ is the sum of (51), (54), and (57), $\Sigma_{n,2}$ is the sum of (52), (55), and (58), and $\Sigma_{n,3}$ is the sum of (53), (56), and (59). One can readily verify that

$$(61) \quad \Sigma_{n,1}(z) = \frac{K_n(z, z)^3}{(1 - |z|^2)^2} - K_n(z, z) |K_n(z, \bar{z})|^2 \left(\frac{2}{(1 - |z|^2)^2} - \frac{1}{|1 - z^2|^2} \right).$$

Furthermore, one can check that the sum of (52) and the first summands of (55) and (58) is equal to

$$2 \frac{\operatorname{Re}((\bar{z} - z) \overline{K_n(z, \bar{z})} (\varphi_n^*(z)^2 - \varphi_n(z)^2) S_n(z, z))}{(1 - |z|^2)^2 |1 - z^2|^2},$$

while the sum of the last summands of (55) and (58) is equal to

$$2 \frac{\operatorname{Re}((z - \bar{z}) \overline{K_n(z, \bar{z})} (|\varphi_n^*(z)|^2 - |\varphi_n(z)|^2) S_n(z, \bar{z}))}{(1 - |z|^2)^2 |1 - z^2|^2}.$$

By adding up the last two expressions and simplifying, we get that

$$(62) \quad \Sigma_{n,2}(z) = \frac{8 \operatorname{Im}(z) \operatorname{Im}(\varphi_n^*(z) \overline{\varphi_n(z)}) \operatorname{Re}((\varphi_n^*)'(z) \varphi_n(z) - \varphi_n'(z) \varphi_n^*(z))}{(1 - |z|^2)^2 |1 - z^2|^2}.$$

To compute $\Sigma_{n,3}$, notice that the sum of the first summands of (53) and (56) is equal to

$$-\frac{K_n(z, z) |(\varphi_n^*)'(z) \varphi_n(z) - \varphi_n'(z) \varphi_n^*(z)|^2}{(1 - |z|^2)^2}.$$

The remaining summand of (53) is equal to

$$(|\varphi'_n(z)|^2 - |(\varphi_n^*)'(z)|^2) \frac{|\varphi_n(z)|^4 + |\varphi_n^*(z)|^4 - 2\operatorname{Re}(\varphi_n^*(z)^2 \overline{\varphi_n(z)^2})}{(1 - |z|^2)|1 - z^2|^2},$$

while the remaining summand of (56) is equal to

$$\begin{aligned} & |\varphi'_n(z)|^2 \frac{|\varphi_n(z)|^4 - |\varphi_n(z)\varphi_n^*(z)|^2}{(1 - |z|^2)|1 - z^2|^2} - |(\varphi_n^*)'(z)|^2 \frac{|\varphi_n^*(z)|^4 - |\varphi_n(z)\varphi_n^*(z)|^2}{(1 - |z|^2)|1 - z^2|^2} \\ & + 2K_n(z, z) \frac{\operatorname{Re}((\varphi_n^*)'(z)\varphi_n^*(z)\overline{\varphi'_n(z)\varphi_n(z)})}{|1 - z^2|^2}. \end{aligned}$$

Moreover, (59) can be rewritten as

$$\begin{aligned} & |\varphi'_n(z)|^2 \frac{2\operatorname{Re}(\varphi_n^*(z)^2 \overline{\varphi_n(z)^2}) - 2|\varphi_n(z)|^4}{(1 - |z|^2)|1 - z^2|^2} - |(\varphi_n^*)'(z)|^2 \frac{2\operatorname{Re}(\varphi_n^*(z)^2 \overline{\varphi_n(z)^2}) - 2|\varphi_n^*(z)|^4}{(1 - |z|^2)|1 - z^2|^2} \\ & - 2K_n(z, z) \frac{\operatorname{Re}((\varphi_n^*)'(z)\varphi_n^*(z)\overline{\varphi'_n(z)\varphi_n(z)} + (\varphi_n^*)'(z)\varphi_n(z)\overline{\varphi'_n(z)\varphi_n^*(z)})}{|1 - z^2|^2}. \end{aligned}$$

By adding the last four expressions together we get that

$$(63) \quad \Sigma_{n,3}(z) = K_n(z, z)|(\varphi_n^*)'(z)\varphi_n(z) - \varphi'_n(z)\varphi_n^*(z)|^2 \left(\frac{1}{|1 - z^2|^2} - \frac{1}{(1 - |z|^2)^2} \right).$$

Notice that

$$(64) \quad \frac{(\varphi_n^*)'(z)\varphi_n(z) - \varphi'_n(z)\varphi_n^*(z)}{\phi_n^2(z)} = \begin{cases} -b'_n(z), & |z| < 1, \\ (b_n^{-1})'(z), & |z| > 1, \end{cases}$$

where

$$\phi_n(z) := \begin{cases} \varphi_n^*(z), & |z| < 1, \\ \varphi_n(z), & |z| > 1. \end{cases}$$

Finally, the assumption $\alpha_i \rightarrow 0$ as $i \rightarrow \infty$ implies that

$$(65) \quad \begin{cases} b_n(z) \rightarrow 0, & \text{locally uniformly in } |z| < 1, \\ b_n^{-1}(z) \rightarrow 0, & \text{locally uniformly in } |z| > 1, \end{cases}$$

as $n \rightarrow \infty$ according to [10, Theorem 1.7.4] and since $b_n^{-1}(z) = b_n(1/z)$. By recalling (20) and pugging (64) into (62), (63) and using (65), we get that

$$(66) \quad \frac{(\Sigma_{n,2} + \Sigma_{n,3})(z)}{|\phi_n(z)|^6} \rightarrow 0$$

as $n \rightarrow \infty$ locally uniformly in $\overline{\mathbb{C}} \setminus \mathbb{T}$. Similarly, we get that

$$(67) \quad \frac{\Sigma_{n,1}(z)}{|\phi_n(z)|^6} \rightarrow \frac{1}{|1 - |z|^2|} \left(\frac{1}{(1 - |z|^2)^2} - \frac{1}{|1 - z^2|^2} \right)^2$$

as $n \rightarrow \infty$ locally uniformly in $\overline{\mathbb{C}} \setminus \mathbb{T}$. Finally, since

$$\frac{K_n(z, z)^2 - |K_n(z, \bar{z})|^2}{|\phi_n(z)|^4} \rightarrow \left(\frac{1}{(1 - |z|^2)^2} - \frac{1}{|1 - z^2|^2} \right)$$

as $n \rightarrow \infty$ locally uniformly in $\overline{\mathbb{C}} \setminus \mathbb{T}$, the claim of the theorem follows from (67), (66), and (60).

5. PROOF OF THEOREM 1.7

Given an arc S , it follows from [8, Theorem 3.1] that

$$(68) \quad \mathbb{E} \left[\left| \frac{1}{n} N_n(\Omega(S, \delta)) - \frac{|S|}{2\pi} \right| \right] \lesssim \frac{1}{\sqrt{n}\delta} \left[\log \sum_{k=0}^{n-1} \mathbb{E}[|\eta_k|] + \max_{0 \leq k \leq n-1} \log \|\varphi_k\|_\infty - \frac{1}{2} \mathbb{E}[\log |D_{n-1}|] \right]^{1/2}$$

with some absolute constant, where $D_n := \eta_n \kappa_n \sum_{k=0}^n \eta_k a_{0,k}$. Since the random variables $\{\eta_k\}$ are standard Gaussian, we have that the first expression in square brackets is of the order $\log n$. Furthermore, it follows from [10, Equation (1.5.17)], the Cauchy-Schwarz inequality, and (11) that

$$\log \|\varphi_k\|_\infty \leq \log \kappa_k + \sum_{j=0}^{k-1} |\alpha_j| \leq \log \kappa_n + \sum_{j=0}^{n-1} |\alpha_j| \leq \log \kappa_n + \left(n \sum_{j=0}^{n-1} \alpha_j^2 \right)^{1/2}.$$

Thus, the middle term on the right-hand side of (68) can be estimated above by

$$\log \kappa_n + \left(n \sum_{j=0}^{n-1} \log \frac{1}{1 - \alpha_j^2} \right)^{1/2} = \log \kappa_n + (2n \log \kappa_n)^{1/2} \lesssim n\sqrt{\varepsilon_n},$$

where we used (11) and (17). Finally, since there is constant C such that $\mathbb{E}[\log |\eta_0 + z|] \geq C$ for all $z \in \mathbb{C}$, it holds that

$$\mathbb{E}[\log |D_n|] = \log \kappa_n + \mathbb{E}[\log |\eta_n|] + \mathbb{E} \left[\log \left| \eta_0 + \sum_{i=1}^{n-1} \eta_i a_{0,i} \right| \right] \geq 2C,$$

(recall also that $\kappa_n \geq 1$ and $a_{0,0} = 1$ as μ is a probability measure). Plugging the above estimates into (68) proves the theorem.

6. PROOF OF THEOREM 1.8

It follows from (4) that

$$\begin{aligned} \frac{1}{n} \mathbb{E}[N(\Omega(S, \tau_1, \tau_2))] &= \frac{1}{n} \iint_{\Omega(S, \tau_1, \tau_2)} \rho_n^{(0,1)}(z) dA \\ &= \frac{1}{2n^2} \int_S \int_{\tau_1}^{\tau_2} \rho_n^{(0,1)} \left(z \left(1 + \frac{\tau}{2n} \right) \right) \left(1 + \frac{\tau}{2n} \right) d\tau |dz|. \end{aligned}$$

Hence, it is enough to show that

$$(69) \quad \frac{1}{2n^2} \rho_n^{(0,1)} \left(z \left(1 + \frac{\tau}{2n} \right) \right) \rightarrow \frac{1}{2\pi} \left(\frac{H'(\tau)}{H(\tau)} \right)'$$

uniformly for $z \in S$ and τ on compact subsets of the real line.

Since $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$, the measure μ is regular in the sense of Ullman-Stahl-Totik, see (11) and (17). Therefore, [4, Theorem 6.3] is applicable on S . That is, it holds that

$$(70) \quad \lim_{n \rightarrow \infty} K_n(z_{n,u}, z_{n,\bar{v}}) K_n^{-1}(z, z) = H(u + v)$$

uniformly for $z \in S$ and u, v on compact subsets of \mathbb{C} , where $z_{n,a} := z(1 + a/n)$. Moreover, we have that

$$\frac{\partial^{i+j}}{\partial u^i \partial v^j} K_n(z_{n,u}, z_{n,\bar{v}}) = \frac{z^{i-j}}{n^{i+j}} K_n^{(i,j)}(z_{n,u}, z_{n,\bar{v}})$$

for any non-negative integers i, j . Thus, it follows from Cauchy's integral formula and (70) that

$$(71) \quad \lim_{n \rightarrow \infty} \frac{z^{i-j}}{n^{i+j}} \frac{K_n^{(i,j)}(z_{n,u}, z_{n,\bar{v}})}{K_n(z, z)} = H^{(i+j)}(u + v)$$

uniformly for $z \in S$ and u, v on compact subsets of \mathbb{C} .

In another connection, since $\alpha_i \rightarrow 0$ as $i \rightarrow \infty$, [7, Theorem 4] states that

$$\lim_{n \rightarrow \infty} \max_{z \in \mathbb{T}} |\varphi_n(z)|^2 K_n^{-1}(z, z) = 0.$$

By compactness, the set S can be covered by finitely many closed subarcs $I_j \subset \mathbb{T} \setminus \{\pm 1\}$ such that μ' is continuous and positive on each I_j . Since $\cup_j I_j$ is separated from ± 1 , the Christoffel-Darboux formula (20) and the above limit yield that

$$\lim_{n \rightarrow \infty} \max_{z \in \cup_j I_j} K_n(z, \bar{z}) K_n^{-1}(z, z) = 0.$$

Since $K_n(z, \bar{z})$ is a polynomial of degree $2n - 2$, it follows from the Bernstein-Walsh inequality (see for example [4, Lemma 6.1]), applied on each I_j separately, that

$$|K_n(z_{n,a}, \bar{z}_{n,a})| \lesssim \max_{z \in \cup_j I_j} |K_n(z, \bar{z})|$$

uniformly in n and a on compact subsets of \mathbb{C} . Thus, it holds that

$$(72) \quad \lim_{n \rightarrow \infty} |K_n(z_{n,a}, \bar{z}_{n,a})| K_n^{-1}(z, z) = 0$$

uniformly for $z \in S$ and a on compact subsets of \mathbb{C} . Moreover, since

$$K_n^{(1,0)}(z_{n,a}, \bar{z}_{n,a}) = \frac{n}{2z} \frac{\partial}{\partial a} K_n(z_{n,a}, \bar{z}_{n,a}),$$

it follows from (72) and Cauchy's integral formula that

$$(73) \quad \lim_{n \rightarrow \infty} n^{-1} |K_n^{(1,0)}(z_{n,a}, \bar{z}_{n,a})| K_n^{-1}(z, z) = 0$$

uniformly for $z \in S$ and a on compact subsets of \mathbb{C} .

The desired claim (69) now is an immediate consequence of (7) and (71)–(73).

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REFERENCES

- [1] V. Andrievskii. Weighted polynomial inequalities in the complex plane. *J. Approx. Theory*, 164:1165–1183, 2012. [8](#), [9](#)
- [2] Ya. L. Geronimus. *Polynomials Orthogonal on a Circle and Their Applications*. Amer. Math. Soc. Translation, Vol. 104, 1954. [11](#)
- [3] M. Kac. On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.*, 49:314–320, 1943. [1](#)
- [4] E. Levin and D. Lubinsky. Universality limits involving orthogonal polynomials on the unit circle. *Comput. Methods Funct. Theory*, 7(2):543–561, 2007. [9](#), [14](#), [15](#)
- [5] D. Lubinsky, I. Pritsker, and X. Xie. Expected number of real zeros for random linear combinations of orthogonal polynomials. *Proc. Amer. Math. Soc.*, 144:1631–1642, 2016. [2](#)
- [6] G. Mastroianni and V. Totik. Weighted polynomial inequalities with doubling and A_∞ weights. *Constr. Approx.*, 16:37–71, 2000. [7](#), [8](#)
- [7] A. Máté, P. Nevai, and V. Totik. Extensions of Szegő's theory of orthogonal polynomials, II *Constr. Approx.*, 3:51–72, 1987. [15](#)
- [8] I. Pritsker and A. Yeager. Zeros of polynomials with random coefficients. *J. Approx. Theory*, 189:88–100, 2015. [4](#), [14](#)
- [9] L.A. Shepp and R.J. Vanderbei. The complex zeros of random polynomials. *Trans. Amer. Math. Soc.*, 347(11):4365–4384, 1995. [1](#)

- [10] B. Simon. *Orthogonal Polynomials on the Unit Circle*. American Mathematical Society Colloquium Publications, Vol. 54, Parts I and II, Providence, RI, 2005. [2](#), [3](#), [4](#), [5](#), [6](#), [7](#), [10](#), [13](#), [14](#)
- [11] R.J. Vanderbei. The Complex zeros of random sums. <http://arxiv.org/abs/1508.05162v1>. [2](#)
- [12] T. Varga. Christoffel functions for doubling measures on quasismooth curves and arcs. *Acta Math. Hungar.*, 141(1):161–184, 2013. [9](#)
- [13] J.E. Wilkins. An asymptotic expansion for the expected number of real zeros of a random polynomials. *Pros. Amer. Math. Soc.*, 103(4):1249–1258, 1988. [1](#)
- [14] M.-L. Wong. A formula for inserting point masses. *J. Comput. and Appl. Math.*, 233:852–855, 2009. [4](#), [11](#)

DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY-PURDUE UNIVERSITY INDIANAPOLIS, 402 NORTH BLACKFORD STREET, INDIANAPOLIS, IN 46202, USA
E-mail address: maxyatts@iupui.edu

DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74078 USA
E-mail address: aaron.yeager@okstate.edu