Optimal Subsampling Approach To The Analysis of Count Data of Massive Size

A Preprint

Fei Tan
Department of Mathematical Sciences, IUPUI
402 N Blackford St., LD 270, Indianapolis, IN 46202, USA
feitan@iupui.edu

Xiaofeng Zhao
School of Mathematics and Statistics
North China University of Water Resources and Electric Power
136 Jinshui Road, Zhengzhou City, Henan Province, P.R. China

Hanxiang Peng
Department of Mathematical Sciences, IUPUI
402 N Blackford St., LD 270, Indianapolis, IN 46202, USA

October 25, 2023

Abstract

The uniform and the nonuniform sampling–the Leverage-Scores (Lev) sampling– are frequently used in the development of fast algorithms. Both approaches, however, are not effective in extracting important information in data. In this article, we propose the A-optimal Subsampling Approach, and give the A-optimal distributions in a generalized linear model (GLM) by minimizing the sum of the component variances of the subsampling estimator. As the A-optimal distributions have the same running time as the full-data estimator, we generalize the Scoring Algorithm in Zhang, *et al.*(2023) in a linear model to GLM using the iterative weighted least squares. The paper presents a comprehensive evaluation of our approach through numerical studies. Our results in simulated and real data indicate that our approach substantially outperforms the uniform and the Lev sampling and the Algorithm significantly reduces the computational time.

Keywords A-optimality · Big Data · Generalized Linear Models · Negative Binomial Regression · Poisson Regression · Zero-inflated Poisson Regression

1 Introduction

Big Data are on a massive scale with regard to volume, velocity, variety, and veracity that exceed both the capacity of the conventional software tools and operating systems and the physical spaces of computers, see e.g. Wang, *et al.* (2015); Fan, *et al.* (2013). Massive data pose two computational bottlenecks: (1) the data exceed a computer's memory, and (2) the computing task requires too long time to finish. The two bottlenecks can be simultaneously addressed by *judiciously* choosing a sub-data as a surrogate for the full data and completing the data analysis. This is the goal that this article will pursue.

While the often used Divide-and-Conquer approach readily breaks the memory limit, the Subsampling approach not only breaks the limit but speed up computing as well as possesses other useful statistical properties. Due to its mathematical simplicity and computational ease, the *uniform sampling* is often used in subsampling for intensive computing and for development of fast randomized algorithms and in re-sampling for Monte Carlo and bootstrap, see Mahoney (2011).

The uniform sampling, however, is not effective in extracting information, see a simulation study in Zhang, *et al.*(2023). In this article, non-uniform sampling distributions will be sought by the criterion of A-optimality, that is, by minimizing the trace of the asymptotic variance-covariance matrix (equivalently, the sum of the component variances) of the subsampling estimator.

Wang, *et al.*(2018) constructed optimal subsampling for large logistic regression. Zhang, *et al.*(2023) studied the A-optimal Subsampling in a Big Data linear regression model. They gave three types of A-optimal sampling distributions, studied the relationship to the leverage-scores-based distribution, discussed truncation and proposed the Scoring Algorithm for fast computation. They proved the almost sure convergence of the subsampling estimator and conducted extensive simulations and real data applications. Our results generalize theirs to GLM.

Mathematicians, computer scientists and statisticians have already made important progress in this area. Drineas, *et al.*(2006a) constructed fast Monte Carlo algorithms to approximate matrix multiplication. Drineas, *et al.*(2006b) presented a sampling algorithm for the least squares fit problem and studied its algorithmic properties. A key feature of the above algorithms is the non-uniform sampling. Ma and Sun (2014) and Ma, *et al.*(2015) used the leverage scores as non-uniform importance sampling distributions for Big Data linear regression. Xu, *et al.*(2016) studied subsampled newton methods with non-uniform sampling. Wang, *et al.*(2019) developed information-based subdata selection for large linear regression. Liang, *et al.*(2013) proposed a resampling-based stochastic approximation for large geostatistical data. Kleiner, *et al.*(2014) gave a scalable bootstrap for massive data. See also the monograph by Mahoney (2011) on nonuniform random subsampling for matrix based machine learning.

Count data are observations of the number of occurrences of a behavior in a fixed period of time. Count data are common, for example, hospital visits, blog comments, car/bike renters, and questionnaire respondents. The analysis of count data is an important task in social sciences and economics. Linear regression is not an appropriate technique for count data, as it fails to take into account the limited number of possible values of the count response variable. Standard regression methods include Poisson, Overdispersed Poisson, Negative Binomial, and Zero-Inflated Poisson regressions, as well as truncated methods and the quasi-likelihood approach.

Poisson regression and Negative Binomial regression are often used in the analysis of count data. It is motivated by the usual consideration for regression analysis, meanwhile, seek to protect and exploit the nonnegative and integer-valued characteristic of the outcome as much as possible. The scope of count data is very wide, including sociology, marketing, demographic economics, accident insurance, manufacturing defects, etc. The analysis of count data has drawn a lot of attention and been an influential part in statistical modeling.

Poisson regression requires distributional assumptions, which restricts its use in reality because real count data usually exhibit over-dispersion, an inflated number of zeros, an absence of certain counts, censoring counts, and missing counts. Overdisperson can be addressed by generalizing Poisson models to, for instance, quasi-Poisson models. Another useful approach is the Negative Binomial model. These models are related to the family of generalized linear models see, e.g., McCullagh and Nelder (1989) and Dobson (2002).

The above models can deal with over-dispersion rather well, but are not enough for modeling excess zeros. To address this, researchers have developed methods for zero-inflated data by including another model component to capture zero counts. This is done by a mixture model that combines a count component with a point mass at zero, see Cameron and Trivedi (2005).

2 Count Data Regression

Let us start with a few models used in our analysis of simulated and real data.

Poisson Model Let Y have Poisson distribution with mean μ, $\mathrm{Poi}(\mu)$, i.e., the probability mass function (pmf) is

$$f_{\mathrm{poi}}(y;\mu) = e^{-\mu}\mu^y/y!, \quad y = 0, 1, 2, \ldots \tag{2.1}$$

The mean and variance are equal, $\mathrm{Var}(Y) = \mu = E(Y)$. In real-life data, however, the equality is usually not met, which is termed as *overdisperson* in the literature.

In the presence of overdispersion, the standard errors (SE) of the estimates in Poisson regression model are deflated, leading to exaggerated test statistic values for parameters and false significant findings accordingly. Overdispersion can often be tested by the usual goodness-of-fit statistic. In our real data analysis, we should perform such tests. An option to handle overdispersion is

Negative Binomial Model Let Y have Negative Binomial with mean μ and overdispersion parameter $\alpha > 0$, $\mathrm{Nb}(\mu, \alpha)$, i.e., the pmf is

$$f_{\mathrm{nb}}(y; \mu, \alpha) = \frac{\Gamma(y + 1/\alpha)}{\Gamma(1/\alpha) y!} (1 + \alpha\mu)^{-1/\alpha} (\mu/(\mu + 1/\alpha))^{-y}, \ y = 0, 1, 2, \dots \tag{2.2}$$

Then $E(Y) = \mu$ and $\mathrm{Var}(Y) = \mu + \alpha\mu^2$ satisfy $\mathrm{Var}(Y) \geq E(Y)$, and $\mathrm{Var}(Y) = E(Y)$ if and only if $\alpha = 0$.

Quasi-likelihood Model Another popular option to handle overdispersion is the *quasi-likelihood model*. This has the advantage of requiring only to specify the mean and variance but not a distribution for the response Y. Specifically, the statistical inference is based on the quasi-likelihood equation,

$$\sum_{i=1}^{n} \frac{y_i - \mu_i(\boldsymbol{\beta})}{V_i(\boldsymbol{\beta}, \phi)} h'(\mathbf{x}_i^\top \boldsymbol{\beta}) \mathbf{x}_i = 0, \tag{2.3}$$

where $\mu_i(\boldsymbol{\beta}) = E(Y_i|\mathbf{x}_i)$ and $V_i(\boldsymbol{\beta}, \phi) = \mathrm{Var}(Y_i|\mathbf{x}_i)$ are the mean and variance functions to be specified, and ϕ is an overdisperson parameter,

The quasi-likelihood model has great flexibility and unifies several models in the sense that the maximum likelihood estimate (MLE) of the models are special cases. Setting $V_i = \mu_i$, Eqt (2.3) gives the MLE of the Poisson model. Setting $V_i = \mu_i(1 + \alpha\mu_i)$ with $\phi = \alpha$, Eqt (2.3) is the estimating equation for the MLE of the Negative Binomial model. Another frequent choice of the variance for overdispersion is $V_i = \phi\mu_i$ with $\phi > 0$. All the three cases can be unified with the form of $V_i = \mu_i + \alpha\mu_i^p$ for $p = 1, 2$.

Zero-Inflated Poisson Model In many real count data, there is an excess of zero counts for which Poisson distribution can not account. Consider a mixture model combining a degenerate distribution at 0 and a Poisson distribution,

$$f_{\mathrm{zip}}(y; \mu, \rho) = \rho f_0(y) + (1 - \rho) f_{\mathrm{poi}}(y; \mu), \quad y = 0, 1, 2, \dots, \tag{2.4}$$

where $f_0(y) = \mathbf{1}[y = 0]$ is the point mass at zero to account for structural zeros. Since

$$f_{\mathrm{zip}}(0; \mu, \rho) = \rho + (1 - \rho) \exp(-\mu),$$

it follows from $0 \leq f_{\mathrm{zip}}(0; \mu, \rho) \leq 1$ that $1/(1 - \exp(\mu)) \leq \rho \leq 1$. This shows that ρ can be negative. A positive ρ represents the probability of structural zeros above the amount of zeros expected under Poisson distribution f_{poi}. A negative ρ means that the amount of zeros is below the expected under the Poisson, and this does not occur very often. The MLE $\hat{\boldsymbol{\beta}}$ can be obtained by solving the generalized estimating equation (GEE),

$$\sum_{i=1}^{n} \frac{f_{\mathrm{poi}}(y_i; \mu_i)}{f_{\mathrm{zip}}(y_i; \mu_i, \rho)} \frac{y_i - \mu_i(\boldsymbol{\beta})}{\mu_i(\boldsymbol{\beta})} h'(\mathbf{x}_i^\top \boldsymbol{\beta}) \mathbf{x}_i = 0. \tag{2.5}$$

To estimate ρ, one can find another equation differentiating the log likelihood with respect to ρ. For simplicity, we shall estimate ρ by the sample percentage $\hat{\rho}$ of the structural zeros. Substituting $\hat{\rho}$ in (2.5), we solve for $\hat{\boldsymbol{\beta}}$.

3 The A-optimal Subsampling In Big Data GLM

In a generalized linear regression model (GLM), the response variable Y_i and covariate vector $\mathbf{x}_i$ satisfy

$$Y_i = h(\mathbf{x}_i^\top \boldsymbol{\beta}) + \varepsilon_i, \quad i = 1, \dots, n, \tag{3.1}$$

where $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown parameter vector, h is the inverse of a link function g, and ε_i's are independent random errors with zero mean $E(\varepsilon_i) = 0$ and finite variance $\mathrm{Var}(\varepsilon_i) = V(\mu_i(\boldsymbol{\beta}_0)) := V_i(\boldsymbol{\beta}_0)$ for some variance function V with the mean $\mu_i(\boldsymbol{\beta}) = E(Y_i) = h(\mathbf{x}_i^\top \boldsymbol{\beta})$ of Y_i. Here $\boldsymbol{\beta}_0$ denotes the true value of $\boldsymbol{\beta}$. If $\mathbf{x}_i$'s are random covariates, we replace these assumptions with the conditional versions given $\mathbf{x}_i$'s.

The parameter $\boldsymbol{\beta}$ can be estimated by the solution $\hat{\boldsymbol{\beta}}$ to the generalized estimating equation (GEE),

$$\sum_{i=1}^{n} \frac{Y_i - \mu_i(\boldsymbol{\beta})}{V_i(\boldsymbol{\beta})} \frac{\mathbf{x}_i}{g_i'(\boldsymbol{\beta})} = 0, \quad g_i'(\boldsymbol{\beta}) = g'(\mathbf{x}_i^\top \boldsymbol{\beta}). \tag{3.2}$$

When n is of massive size (often accompanied with large p), it becomes a challenging task to compute the usual $\hat{\boldsymbol{\beta}}$ using the conventional computers and software tools. We now take a random subsample $(\mathbf{X}^*, \mathbf{Y}^*)$ of size $r << n$ as surrogate and construct a subsampling estimator $\hat{\boldsymbol{\beta}}_r^*$ in Fig. 1 to approximate $\hat{\boldsymbol{\beta}}$.

Figure 1: The Subsampling Estimator $\hat{\boldsymbol{\beta}}_r^*$

1. Construct a distribution $\boldsymbol{\pi}$ on the data points $(\mathbf{x}_i, Y_i)$'s, use it to draw a subsample $(\mathbf{X}^*, \mathbf{Y}^*)$ of size $r << n$ and formulate the diagonal matrix $\mathbf{W}^* = \text{diag}(1/r\boldsymbol{\pi}^*)$ with $\boldsymbol{\pi}^*$ the corresponding probability vector.
2. Calculate the subsampling estimator $\hat{\boldsymbol{\beta}}_r^*$ as a solution to the GEE,
$$\sum_{j=1}^{r} \frac{Y_j^* - \mu_j^*(\boldsymbol{\beta})}{\pi_j^* V_j^*(\boldsymbol{\beta})} \frac{\mathbf{x}_j^*}{g_j'^*(\boldsymbol{\beta})} = 0, \tag{3.4}$$
where $\mu_j^*(\boldsymbol{\beta}) = h(\mathbf{x}_j^{*\top}\boldsymbol{\beta}), V_j^*(\boldsymbol{\beta}) = V(\mu_j^*(\boldsymbol{\beta}))$ and $g_j'^*(\boldsymbol{\beta}) = g_j'(\mu_j^*(\boldsymbol{\beta}))$.

Notation Set $\varepsilon_i(\boldsymbol{\beta}) = Y_i - \mu_i(\boldsymbol{\beta})$, $V_i(\boldsymbol{\beta}) = \text{Var}(\varepsilon_i(\boldsymbol{\beta}))$, $\Sigma(\boldsymbol{\beta}) = \text{Diag}(V_i(\boldsymbol{\beta})) > 0$, $\dot{g}(m) = g'(m)$, $\mu_i = \mu_i(\boldsymbol{\beta}_0)$, $V_i = V_i(\boldsymbol{\beta}_0)$, $g_i = g_i(\boldsymbol{\beta}_0)$, $\varepsilon_i = Y_i - \mu_i$, $\Sigma = \text{Diag}(V_i)$, $\hat{\mu}_i = \mu_i(\hat{\boldsymbol{\beta}})$, $\hat{\varepsilon}_i = Y_i - \hat{\mu}_i$, $\hat{V}_i = V_i(\hat{\boldsymbol{\beta}})$, $\hat{g}_i = g_i(\hat{\boldsymbol{\beta}})$, and $\hat{\Sigma} = \Sigma(\hat{\boldsymbol{\beta}})$. Denote by $\tilde{\eta}$ the "standarization" of η such as $\tilde{\varepsilon}_i = (Y_i - \mu_i)/\sqrt{V_i}$, and the "hat" version $\hat{\tilde{\varepsilon}}_i = (Y_i - \hat{\mu}_i)/\sqrt{\hat{V}_i}$ of $\tilde{\varepsilon}_i$.

In GLM, the hat matrix is defined as $\mathbf{H}(\boldsymbol{\beta}) = \Sigma^{1/2}(\boldsymbol{\beta})\mathbf{X}(\mathbf{X}^\top\Sigma(\boldsymbol{\beta})\mathbf{X})^{-1}\mathbf{X}^\top\Sigma^{1/2}(\boldsymbol{\beta})$. As $\mathbf{H} = \mathbf{H}(\boldsymbol{\beta}_0)$ contains the unknown parameter $\boldsymbol{\beta}_0$, it can be estimated by the plug-in estimator $\hat{\mathbf{H}} = \mathbf{H}(\hat{\boldsymbol{\beta}})$. When Σ is the identity matrix, $\mathbf{H}$ is the hat matrix in a linear regression model. The hat matrix $\mathbf{H}$ in GLM possesses similar properties as the hat matrix in linear regression models. Like in a linear model, the diagonal entries $h_{i,i}$ of $\mathbf{H}$ induce an importance sampling distribution $\boldsymbol{\ell} = (\ell_i)$ as follows:
$$\ell_i \propto h_{i,i}, \quad i = 1, \ldots, n, \tag{3.3}$$
where $b_i \propto c_i$ for all i denotes $b_i = c_i/\sum_j c_j$. Obviously, $\ell_i = h_{i,i}/p$ as in a linear regression model.

Under suitable conditions, the subsampling estimator $\hat{\boldsymbol{\beta}}_r^*$ is asymptotically normal,
$$\mathbf{V}_0^{-1/2}(\boldsymbol{\pi})\sqrt{r}(\hat{\boldsymbol{\beta}}_r^* - \hat{\boldsymbol{\beta}}) \Rightarrow N(0, \mathbf{I}_p), \quad \text{a.s.} \quad r \to \infty. \tag{3.5}$$
where $\mathbf{V}_0(\boldsymbol{\pi})$ is the asymptotic variance-covariance matrix of $\hat{\boldsymbol{\beta}}_r^*$, given by
$$\mathbf{V}_0(\boldsymbol{\pi}) = (\mathbf{X}^\top\hat{\Sigma}\mathbf{X})^{-1}(\hat{\Sigma}^{1/2}\mathbf{X})^\top\text{Diag}(\hat{\tilde{\varepsilon}}^2/r\boldsymbol{\pi})\hat{\Sigma}^{1/2}\mathbf{X}(\mathbf{X}^\top\hat{\Sigma}\mathbf{X})^{-1}. \tag{3.6}$$

Let $\mathbf{A}$ be a nonsingular $q \times p$ matrix. The plug-in estimator $\mathbf{A}\hat{\boldsymbol{\beta}}_r^*$ of the linear transformation $\mathbf{A}\boldsymbol{\beta}$ of $\boldsymbol{\beta}$ then has the asymptotic variance-covariance matrix $\mathbf{A}\mathbf{V}_0(\boldsymbol{\pi})\mathbf{A}^\top$. The criteria of A-optimality is to seek a sampling distribution $\boldsymbol{\pi}$ supported on the data points $\{(\mathbf{x}_i, Y_i)\}$ which minimizes the trace $\text{Tr}(\mathbf{A}\mathbf{V}_0(\boldsymbol{\pi})\mathbf{A}^\top)$ of the matrix. Equivalently, the criteria seeks $\boldsymbol{\pi}$ to minimize the sum of the variances of the components of $\mathbf{A}\hat{\boldsymbol{\beta}}_r^*$. One has
$$\hat{\tau}_{\mathbf{A}}(\boldsymbol{\pi}) =: \text{Tr}(\mathbf{A}\mathbf{V}_0(\boldsymbol{\pi})\mathbf{A}^\top) = \frac{1}{r}\sum_{i=1}^{n} \frac{\|\mathbf{a}_i\|^2\hat{\tilde{\varepsilon}}_i^2}{\pi_i}, \tag{3.7}$$
where $\mathbf{a}_i = \mathbf{A}(\mathbf{X}^\top\hat{\Sigma}\mathbf{X})^{-1}\hat{\Sigma}^{1/2}\mathbf{x}_i$. Using Lagrange's multiplier method, we derive

Theorem 1 *Suppose that $\mathbf{A}$ is independent of $\boldsymbol{\pi}$. Assume that $\mathbf{X}^\top\hat{\Sigma}\mathbf{X}$ is invertible such that $\mathbf{A}(\mathbf{X}^\top\hat{\Sigma}\mathbf{X})^{-1}\hat{\Sigma}^{1/2}\mathbf{x}_i \neq 0$ and the diagonal entries $\hat{h}_{i,i}$ of $\hat{\mathbf{H}}$ satisfy $\hat{h}_{i,i} \neq 1$ for $i = 1, \ldots, n$. Then there exists a unique A-optimal distribution $\hat{\boldsymbol{\pi}}_{\mathbf{A}} = (\hat{\pi}_i^{(\mathbf{A})})$ for $\mathbf{A}\hat{\boldsymbol{\beta}}_r^*$ to approximate the best linear unbiased estimate $\mathbf{A}\hat{\boldsymbol{\beta}}$ of $\mathbf{A}\boldsymbol{\beta}$, which is given by*
$$\hat{\pi}_i^{(\mathbf{A})} \propto \|\mathbf{a}_i\|\,|\hat{\tilde{\varepsilon}}_i|, \quad i = 1, \ldots, n. \tag{3.8}$$

Let $\mathbf{H}_\alpha = (h_{\alpha,i,j}) = \Sigma^{1/2}\mathbf{X}(\mathbf{X}^\top\Sigma\mathbf{X})^{-\alpha}\mathbf{X}^\top\Sigma^{1/2}, \alpha \in \mathbb{R}$. Obviously, $\mathbf{H}_1 = \mathbf{H}$ and $\mathbf{H}_0 = \Sigma^{1/2}\mathbf{X}\mathbf{X}^\top\Sigma^{1/2}$. For $\mathbf{A} = (\mathbf{X}^\top\Sigma\mathbf{X})^{1-\alpha/2}$, one obtains $\hat{\boldsymbol{\pi}}_\alpha = \hat{\boldsymbol{\pi}}^{(\mathbf{A})}$. The A-optimal distribution $\hat{\boldsymbol{\pi}}_{\text{aopt}} = (\hat{\pi}_{2,i})$ for $\hat{\boldsymbol{\beta}}_r^*$ to approximate $\hat{\boldsymbol{\beta}}$ is the case of $\alpha = 2$. Specifically,
$$\hat{\pi}_i^{(\text{aopt})} =: \hat{\pi}_{2,i} \propto \hat{h}_{2,i,i}^{1/2}\,|\hat{\tilde{\varepsilon}}_i|, \quad i = 1, 2, \ldots, n. \tag{3.9}$$

Another two A-optimal sampling distributions of possibly computational ease are
$$\hat{\pi}_{0,i} \propto \hat{h}_{0,i,i}^{1/2}\,|\hat{\tilde{\varepsilon}}_i| = \|\hat{V}_i^{1/2}\mathbf{x}_i\|\,|\hat{\tilde{\varepsilon}}_i|, \quad \hat{\pi}_{1,i} \propto \hat{h}_{i,i}\,|\hat{\tilde{\varepsilon}}_i|, \quad i = 1, \ldots, n. \tag{3.10}$$

In our analysis of simulated and real data, the Poisson (Poi), the Negative Binomial (NB) and the Quasipoisson (QPoi) models were used with the log-link $g(m) = \log(m)$, so that $\hat{\mu}_i = \exp(\mathbf{x}_i^\top \hat{\boldsymbol{\beta}})$ and $\hat{\hat{\varepsilon}}_i = (Y_i - \hat{\mu}_i)/\hat{V}_i^{1/2}$ with $\hat{V}_i$ equal to

$$\hat{\mu}_i(\text{Poi}), \quad \hat{\mu}_i(1+\hat{\alpha}\hat{\mu}_i)(\text{NB}), \quad \hat{\phi}\hat{\mu}_i(\text{QPoi}), \quad i = 1, 2, \ldots, n, \tag{3.11}$$

where $\hat{\phi}$ and $\hat{\alpha}$ are estimates of ϕ and α such as the empirical estimators using the full sample in our real data analysis.

A-optimal Sampling via Conditioning Consider minimizing the trace norm of the conditional variance-covariance matrix given $\mathbf{X}$. Note that $\hat{\tau}_{\mathbf{A}}(\boldsymbol{\pi})$ in (3.7) depends on $\hat{\boldsymbol{\beta}}$, $\hat{\tau}_{\mathbf{A}}(\boldsymbol{\pi}) = \hat{\tau}_{\mathbf{A}}(\hat{\boldsymbol{\beta}}, \boldsymbol{\pi})$. Let $\tilde{\tau}_{\mathbf{A}}(\boldsymbol{\pi}) = \mathrm{E}(\hat{\tau}_{\mathbf{A}}(\boldsymbol{\beta}_0, \boldsymbol{\pi})|\mathbf{X})$. We integrate out the standardized squared residuals in $\tilde{\tau}_{\mathbf{A}}(\boldsymbol{\pi})$ to get $\mathrm{Var}(\tilde{\varepsilon}_i) = 1$, so that typically

$$\tilde{\tau}_{\mathbf{A}}(\boldsymbol{\pi}) = \frac{1}{r}\sum_{i=1}^{n} \frac{\|\mathbf{a}_{0,i}\|^2}{\pi_i} \approx \frac{1}{r}\sum_{i=1}^{n} \frac{\|\mathbf{a}_i\|^2}{\pi_i} =: \bar{\tau}_{\mathbf{A}}(\boldsymbol{\pi}), \tag{3.12}$$

where $\mathbf{a}_{0,i} = \mathbf{A}(\mathbf{X}^\top \Sigma \mathbf{X})^{-1}\Sigma^{1/2}\mathbf{x}_i$. Analogously, we minimize $\bar{\tau}(\boldsymbol{\pi})$ and obtain

Theorem 2 *Assume the assumptions in Theorem 1. Then there exists a unique $\bar{A}$-optimal distribution $\bar{\boldsymbol{\pi}}_{\mathbf{A}}$ for $\mathbf{A}\hat{\boldsymbol{\beta}}_r^*$ to approximate the BLUE $\mathbf{A}\hat{\boldsymbol{\beta}}$ of the linear transformation $\mathbf{A}\boldsymbol{\beta}$, given by*

$$\bar{\pi}_i^{(\mathbf{A})} \propto \|\mathbf{a}_i\|, \quad i = 1, \ldots, n. \tag{3.13}$$

The $\bar{A}$-optimal distribution $\bar{\boldsymbol{\pi}}_{\text{aopt}} = (\bar{\pi}_i^{(\text{aopt})})$ for $\hat{\boldsymbol{\beta}}_r^*$ to approximate the estimator $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ is now given by

$$\bar{\pi}_i^{(\text{aopt})} =: \bar{\pi}_{2,i} \propto \sqrt{h_{2,i,i}}, \quad i = 1, \ldots, n. \tag{3.14}$$

Two $\bar{A}$-optimal distributions of possibly computational ease are

$$\bar{\pi}_{0,i} \propto \sqrt{h_{0,i,i}} = \|\hat{V}_i^{1/2}\mathbf{x}_i\|, \quad \bar{\pi}_{1,i} \propto \sqrt{h_{i,i}}, \quad i = 1, \ldots, n. \tag{3.15}$$

Truncation. Observe that (3.8) implies that the i-th data point $(\mathbf{x}_i, Y_i)$ must be drawn with probability $\hat{\pi}_i^{(\mathbf{A})}$ proportional to the i-th standardized residual $|\hat{\hat{\varepsilon}}_i|$. Since each probability is inversely used in constructing $\hat{\boldsymbol{\beta}}_r^*$, $\hat{\pi}_{\mathbf{A}}$ must be truncated from below in order to guarantee appropriate statistical properties for $\hat{\boldsymbol{\beta}}_r^*$. Here we follow Zhang, *et al.*(2023) and truncate $\hat{\boldsymbol{\pi}}_{\mathbf{A}}$ from below by L/n and define $\hat{\boldsymbol{\pi}}_{\mathbf{A}}(l)$ by

$$\hat{\pi}_i^{(\mathbf{A})}(l) \propto \hat{\pi}_{\mathbf{A},i}\mathbf{1}[\hat{\pi}_{\mathbf{A},i} \geq L/n] + (l/n)\mathbf{1}[\hat{\pi}_{\mathbf{A},i} < L/n], \quad i = 1, 2, \ldots, n,$$

where L is a threshold value, and typically $0 < L \leq 1$. As pointed out by the above authors, we may drop "unimportant" observations by taking $l = 0$ for fast computing, otherwise $l = L$. To determine the value of L, we must take it into consideration the desired running time and the accuracy. Our extensive simulations and real data applications exhibited that truncation led to only a slight loss of efficiency.

The Scoring Algorithm. Like a typical optimal sampling, the A-optimal sampling distributions $\hat{\boldsymbol{\pi}}_k$ and $\bar{\boldsymbol{\pi}}_k, k = 0, 1, 2$ have the same running time as the full data estimator $\hat{\boldsymbol{\beta}}$. Here we generalize it to GLM the Scoring Alogrithm proposed for the A-optimal Subsampling in a linear regression model by Zhang, *et al.*(2023).

One advantage of GLM is that the estimator $\hat{\boldsymbol{\beta}}$ can be found by an iterative weighted least squares estimate (IWLSE). Specifically, we rewrite (3.2) in matrix form,

$$\mathbf{X}^\top \mathbf{W}(\boldsymbol{\beta})\dot{\mathbf{g}}(\boldsymbol{\beta})(\mathbf{Y} - \boldsymbol{\mu}(\boldsymbol{\beta})) = 0, \tag{3.16}$$

where $\boldsymbol{\mu}(\boldsymbol{\beta}) = (\mu_1, \ldots, \mu_n)^\top|_{\boldsymbol{\beta}}$, $\dot{\mathbf{g}}(m) = \mathrm{Diag}(g_1', \ldots, g_n')|_m$ and $\mathbf{W}(\boldsymbol{\beta}) = \mathrm{Diag}(1/V_1 g_1^{'2}, \ldots, 1/V_n g_n^{'2})|_{\boldsymbol{\beta}}$. Let

$$\mathbf{Z}^{(0)} = \mathbf{X}\boldsymbol{\beta}^{(0)} + \dot{\mathbf{g}}^{(0)}(\mathbf{Y} - \boldsymbol{\mu}^{(0)}),$$

where $\boldsymbol{\beta}^{(0)}$ is an initial value (which is automatically provided in the *R* package), $\mathbf{W}^{(0)} = \mathbf{W}(\boldsymbol{\beta}^{(0)})$, $\dot{\mathbf{g}}^{(0)} = \dot{\mathbf{g}}(\boldsymbol{\beta}^{(0)})$ and $\boldsymbol{\mu}^{(0)} = \boldsymbol{\mu}(\boldsymbol{\beta}^{(0)})$. The estimator $\hat{\boldsymbol{\beta}}$ can now be obtained by a few iterations of the WLSE,

$$\boldsymbol{\beta}^{(1)} = (\mathbf{X}^\top \mathbf{W}^{(0)}\mathbf{X})^{-1}\mathbf{X}^\top \mathbf{W}^{(0)}\mathbf{Z}^{(0)}. \tag{3.17}$$

Since the computational bottleneck is to invert $\mathbf{X}^\top \mathbf{W}^{(0)}\mathbf{X}$, we shall approximate it by the subsampling matrix $(\mathbf{X}_0^{*\top}\mathbf{W}_0^{(0)*}\mathbf{X}_0^*)^{-1}$ based on a computationally easy pre-subsample $(\mathbf{X}_0^*, \mathbf{y}_0^*)$ from the full data $(\mathbf{X}, \mathbf{Y})$. This is of course implmented by the IWLSE. Denote the resulting estimate and the residual vector by

$$\hat{\boldsymbol{\beta}}_0^* = (\mathbf{X}_0^{*\top}\mathbf{W}_0^{(0)*}\mathbf{X}_0^*)^{-1}\mathbf{X}_0^{*\top}\mathbf{W}_0^{(0)*}\mathbf{Z}_0^{(0)*}, \quad \hat{\boldsymbol{\varepsilon}}_0^* = \mathbf{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}_0^*,$$

where $(\mathbf{X}_1, \mathbf{y}_1)$ is the remaining observations in $(\mathbf{X}, \mathbf{y})$. Compute $\Sigma_0^* = \Sigma(\hat{\boldsymbol{\beta}}_0^*)$ and

$$\mathbf{H}_{0,\alpha}^* = (\Sigma_0^*)^{1/2}\mathbf{X}_1(\mathbf{X}_0^{*\top}\Sigma_0^*\mathbf{X}_0^*)^{-\alpha}\mathbf{X}_1^\top(\Sigma_0^*)^{1/2}, \quad \alpha = 0, 1, 2. \tag{3.18}$$

Our extensive simulations and real data applications in Sections 4–5 exhibited that the Scoring Algorithm worked well.

Figure 2: The Scoring Algorithm

1. Take a uniform pre-subsample $(\mathbf{X}_0^*, \mathbf{y}_0^*)$ of size $r_0 \ll r \ll n$ from $(\mathbf{X}, \mathbf{y})$, and use it to compute $\mathbf{H}_{0,\alpha}^*$ and $\hat{\varepsilon}_0^*$ and the sampling distribution $\boldsymbol{\pi}^0$.
2. Call Algorithm 1 in Fig. 1 with the subsample size r and the A-optimal sampling distribution $\boldsymbol{\pi}^0$.

4 A Large Simulation Study

In this section, we report some simulation results about the A-optimal Subsampling approach. The covariate $\mathbf{X}$ was generated from one of the four multivariate distributions. (GA) The Gaussian $N(0, \Sigma)$ with $\Sigma_{i,j} = 0.3^{|i-j|}$; (MG) The Mixture Gaussian $\frac{1}{2}N(0, \Sigma) + \frac{1}{2}N(0, 3\Sigma)$; (LN) The Log-normal $LN(0, \frac{1}{2}\Sigma)$; (T_5) The student t with 5 degrees of freedom $\mathbf{T}_5(0, \frac{1}{2}\Sigma)$. The response Y_i was generated from Poisson distribution and the Negative Binomial with the variance structure $\mathrm{Var}(Y_i) = \mu_i + 5\mu_i^2$. We chose $n = 50,000$, $p = 50$, and $\boldsymbol{\beta}_0 = (0.1, -0.1 \times \mathbf{1}_{(p/2)}^\top, 0.1 \times \mathbf{1}_{(p/2)}^\top)$.

For the Poisson and the Negative Binomial models, we used each of the $\hat{A}$-optimal distributions given in (3.9)–(3.10) with the estimates in (3.11) and the $\bar{A}$-optimal ones in (3.14)-(3.15). A subsample of size r was then drawn, and the subsampling estimator $\hat{\boldsymbol{\beta}}_r^*$ calculated as in Fig. 1. We computed the empirical mean squared errors for efficiency comparison as follows:

$$\mathrm{MSE} = \frac{1}{B}\sum_{b=1}^{B} \|\hat{\boldsymbol{\beta}}_{r,b}^* - \hat{\boldsymbol{\beta}}\|^2, \quad B = 1,000,$$

where $\hat{\boldsymbol{\beta}}_{r,b}^*$ is the estimator based on the b^{th} subsample of size r.

Variability of $\hat{A}$- and $\bar{A}$-optimality Reported in Fig. 3–4 are the boxplots of the probabilities of six optimal sampling distributions against covariate $\mathbf{X}$ generated from four distributions using Poisson and Negative Binomial regression models. In each plot, all $\hat{\boldsymbol{\pi}}^{(k)}$ were more spread out than all $\bar{\boldsymbol{\pi}}^{(k)}$, but the median of $\bar{\boldsymbol{\pi}}^{(k)}$ is slightly bigger than that of $\hat{\boldsymbol{\pi}}^{(k)}$ for $k = 0, 1, 2$.

MSE Reported in Fig. (5)-(6) are the plots of the logarithm of the MSE of $\hat{\boldsymbol{\beta}}_r^*$ against subsmaple r in Poisson and Negative Binomial regression models, respectively. For the four datasets, the MSE values decreased with the increasing r. Both $\hat{\boldsymbol{\pi}}^{(k)}$ and $\bar{\boldsymbol{\pi}}^{(k)}$ had smaller MSE than the uniform; the $\hat{A}$-sampling outperformed the $\bar{A}$-sampling; $\hat{\boldsymbol{\pi}}^{(2)}$ was the best among $\hat{\boldsymbol{\pi}}^{(k)}$, and $\bar{\boldsymbol{\pi}}^{(2)}$ was the best among $\bar{\boldsymbol{\pi}}^{(k)}, k = 0, 1, 2$.

Coverage Probability Reported on Fig. 7–8 are the plots of the simulated percentages of the 95% confidence intervals catching the true value of the coefficient β_2 against subsample size r based on $2,000$ repetitions. The confidence interval was calculated by $\hat{\beta}_{2,r}^* \pm Z_{0.975} SE(\hat{\beta}_{2,r}^*)$ with $SE(\hat{\beta}_{2,r}^*) = \sqrt{\hat{\mathbf{V}}_{22}}$. Fig. 7 indicated that when subsample size r was small, the coverage probabilities were lower than the nominal level 95%, and were closer to the nominal level with the increasing r. Except for GA and LN, the coverage probabilities under $\hat{\boldsymbol{\pi}}^{(2)}$ and $\bar{\boldsymbol{\pi}}^{(2)}$ were closer to the nominal level than the uniform.

MSE Ratio Reported on Tables (1)–(6) are the ratios of the MSE of the subsampling estimators using the optimal sampling distributions to the MSE of that using the uniform. First, all the ratios in the Tables were less than one, indicating all the optimal subsampling distributions outperformed the uniform. Second, $\hat{\boldsymbol{\pi}}^{(k)}$ outperformed $\bar{\boldsymbol{\pi}}^{(k)}$; and $\hat{\boldsymbol{\pi}}^{(2)}$ was the superior to all others. Third, the simulated MSE ratios using the truncated $\hat{\boldsymbol{\pi}}^{(k)}$ and $\bar{\boldsymbol{\pi}}^{(k)}$ resulted in only slight loss of efficiency compared to those using the untruncated ones for $k = 0, 1, 2$. This property is useful in the Analysis of Big Data as the loss of efficiency would be small when one drops those observations whose sampling probabilities are smaller than certain threshold value for fast computation. Truncation is necessary in order to guarantee the theoretical properties of the proposed procedure, see Zhang, *et al.*(2023).

Reported on Tables (7)–(8) are the MSE ratios where the Scoring Algorithm was used. We first chose a uniform pre-subsample of size $r_0 = 500$; obtained an initial estimator $\hat{\boldsymbol{\beta}}_{r_0}^*$ to approximate $\hat{\boldsymbol{\beta}}$; then approximated the subsampling probabilities and used them to draw subsamples; calculated the subsampling estimator $\hat{\boldsymbol{\beta}}_r^*$ in the end. One observes that the Scoring Algorithm saved significant time while the loss of efficiency was marginal.

Running Time In order to evaluate the computational efficiency, we report the running times for computing $\hat{\boldsymbol{\beta}}_r^*$ by using $\hat{\boldsymbol{\pi}}^{(2)}$ and $\bar{\boldsymbol{\pi}}^{(2)}$ in Tables (9)–(10). The simulation was carried out using the R programming language. Those values were computed on a desktop with Intel i5 processor and 8GB memory. We recorded the CPU times for 1000 repetitions, then took the average of the times for fair comparison. We observe that $\hat{\pi}^{(2)}$ required more time than $\bar{\pi}^{(2)}$. All the proposed methods had significant less computing times than using the full data. In Table (11), we can see all the proposed methods had similar number of iterations, indicating smaller subsample sizes did not necessarily increase the iterations in using Newton's method.

Figure 3: The boxplots of log(probabilities) of the A-optimal sampling distributions for the simulated data in the Poisson model using the full-sample estimator $\hat{\boldsymbol{\beta}}$ for $n = 50,000$ and $p = 50$.

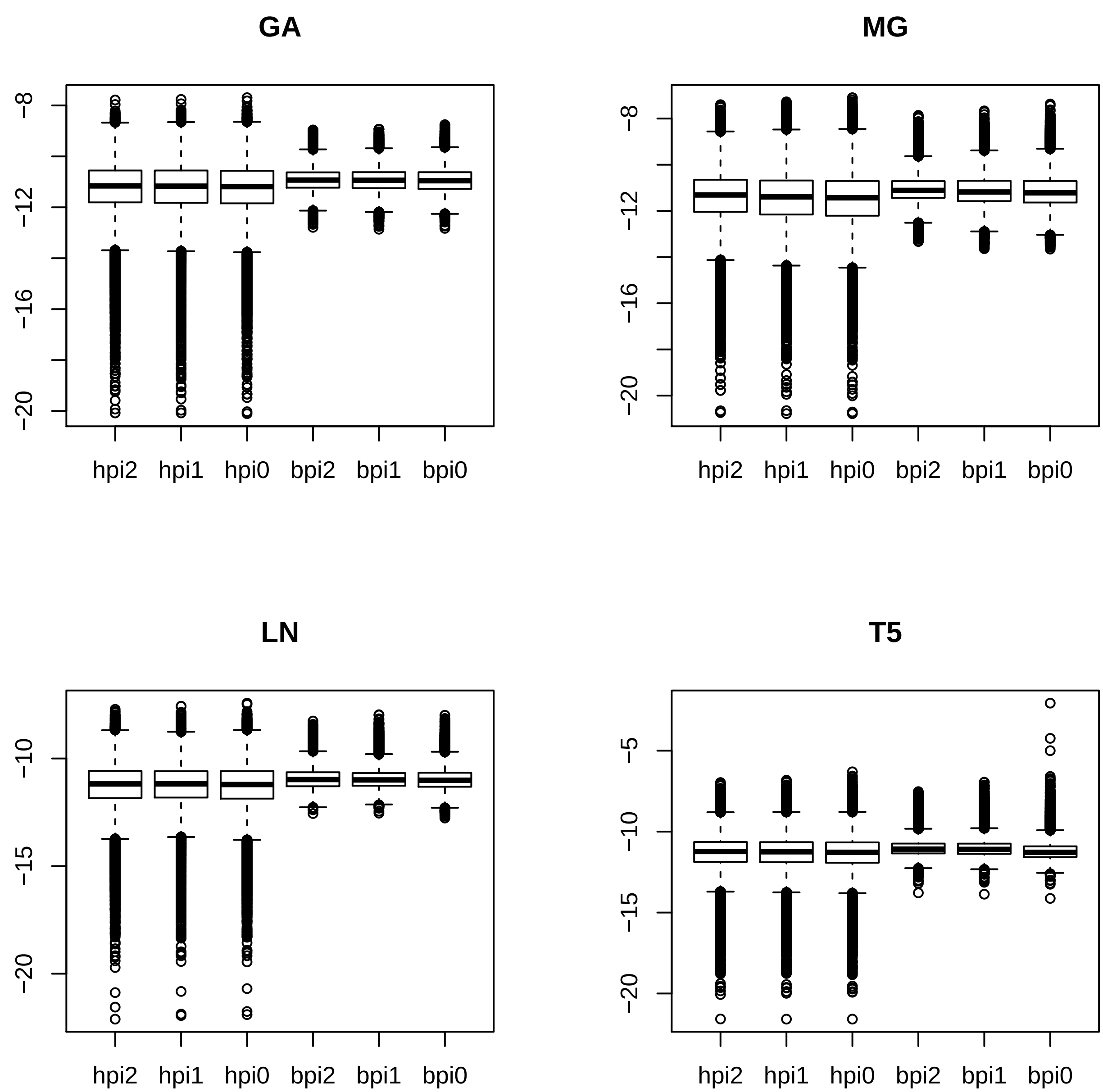


Figure 4: Same as Fig 3 except for the Negative Binomial model.

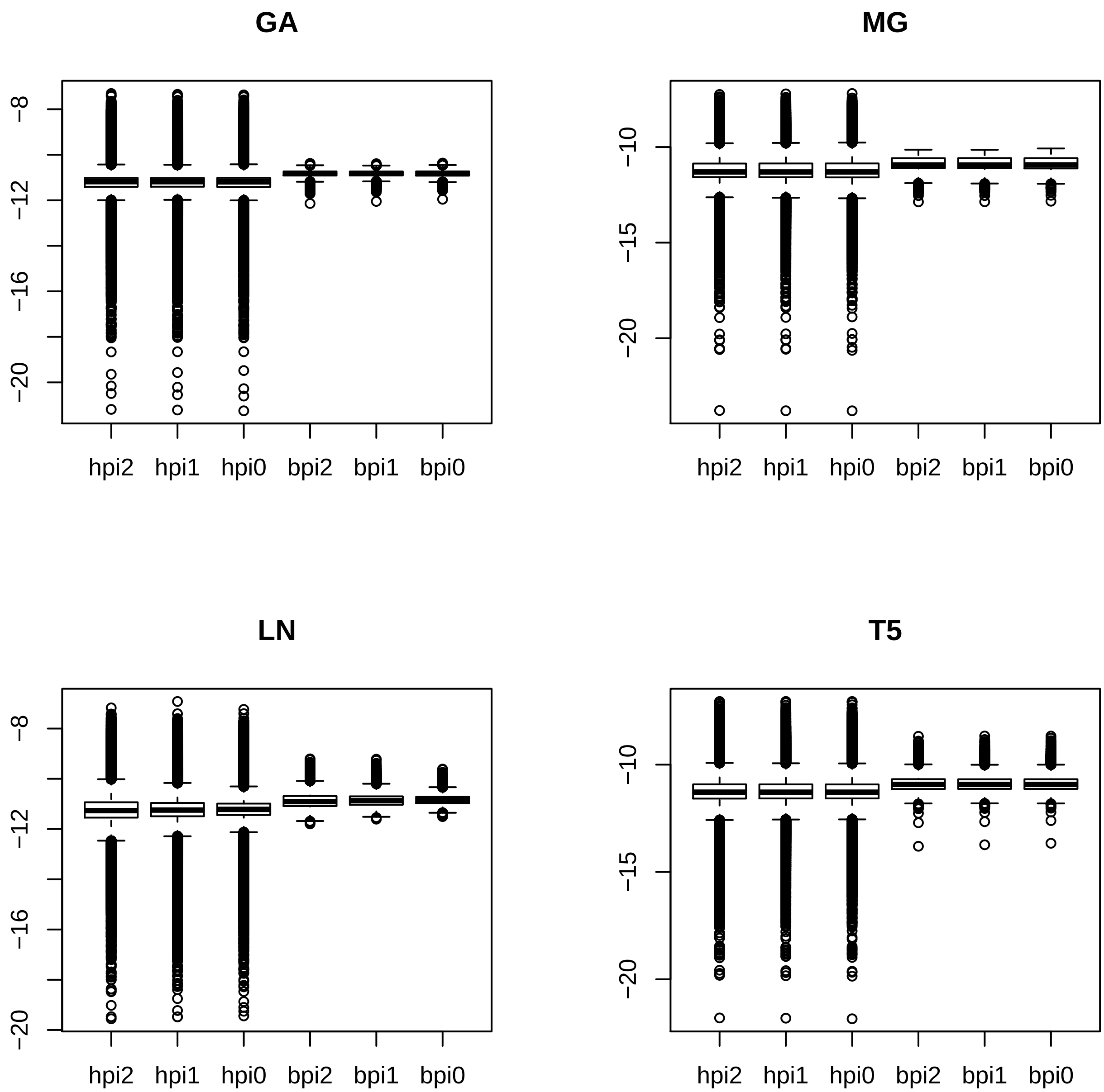


Figure 5: The plots of the logarithm of the MSE of the subsampling estimator $\hat{\boldsymbol{\beta}}^*$ using different sampling distributions against subsample size r in the Poisson model using the full-sample estimator $\hat{\boldsymbol{\beta}}$ for $n = 50,000$ and $p = 50$.

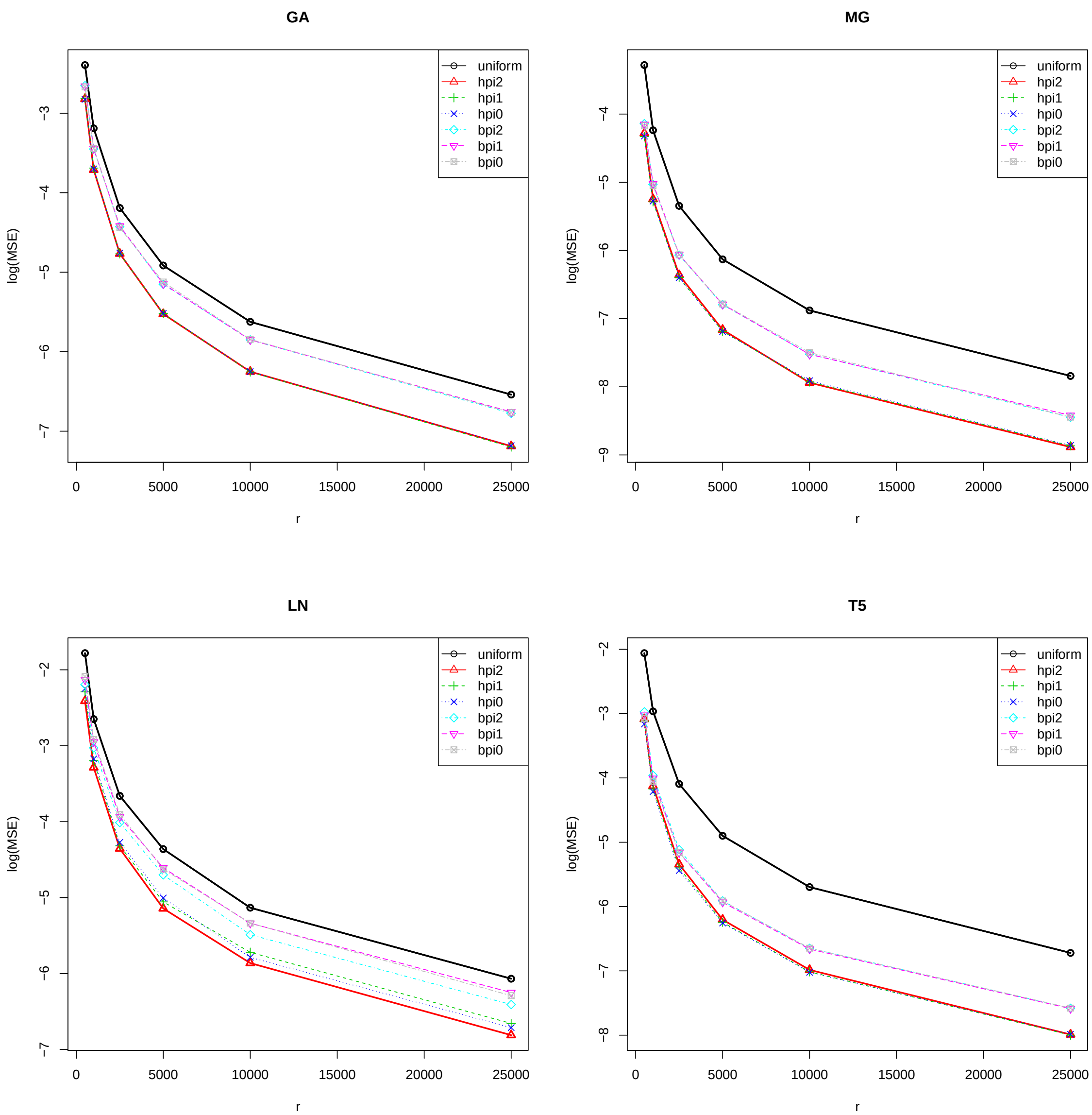


Figure 6: Same as Fig 5 except for the Negative Binomial model.

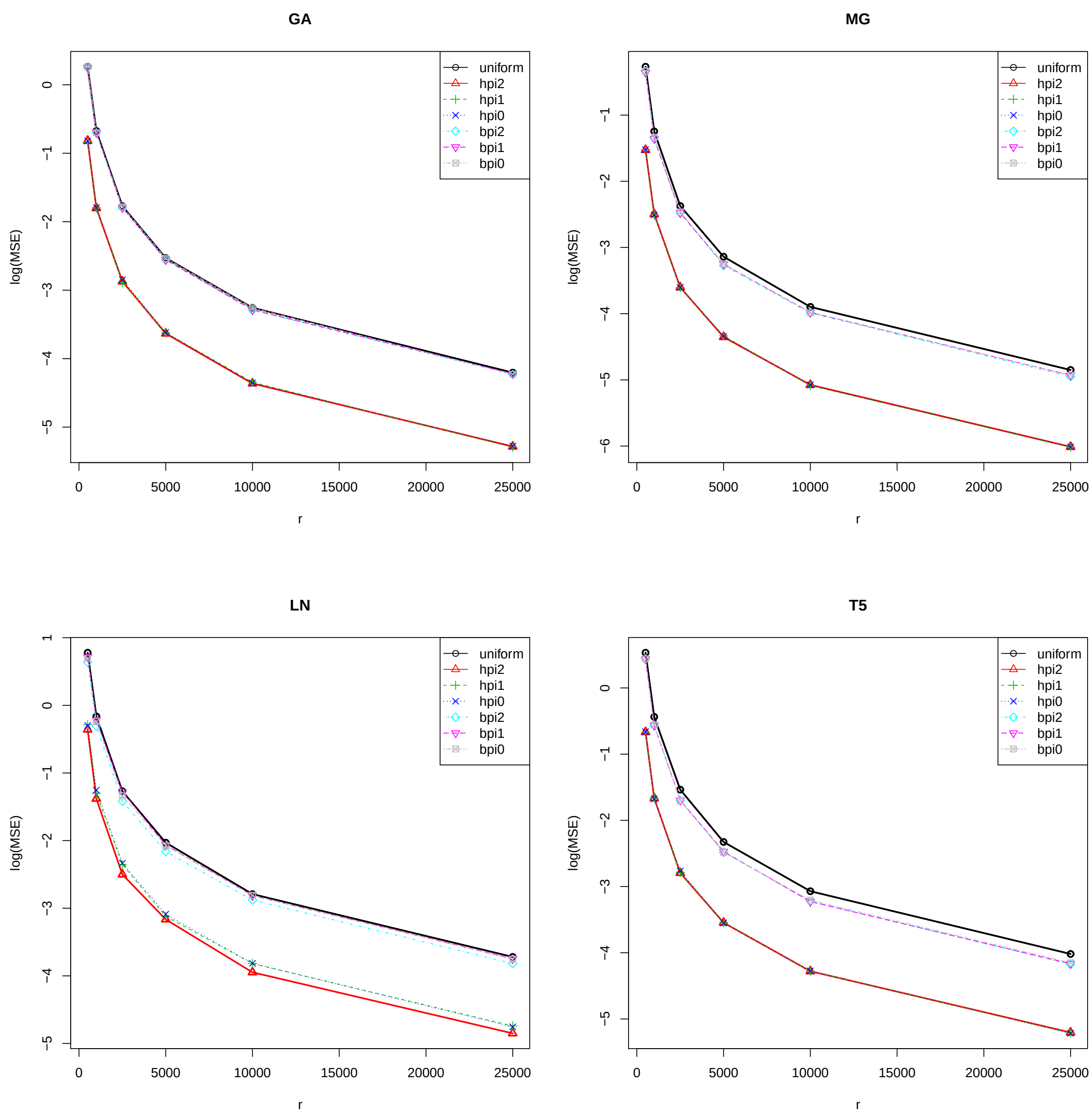


Figure 7: Simulated percentages of the 95% confidence intervals catching the true coefficient β_2 against r in the Poisson model for pre-subsample size $r_0 = 500$, $n = 50,000$ and $p = 50$ using $\hat{\boldsymbol{\pi}}^{(2)}$, $\bar{\boldsymbol{\pi}}^{(2)}$ and the uniform.

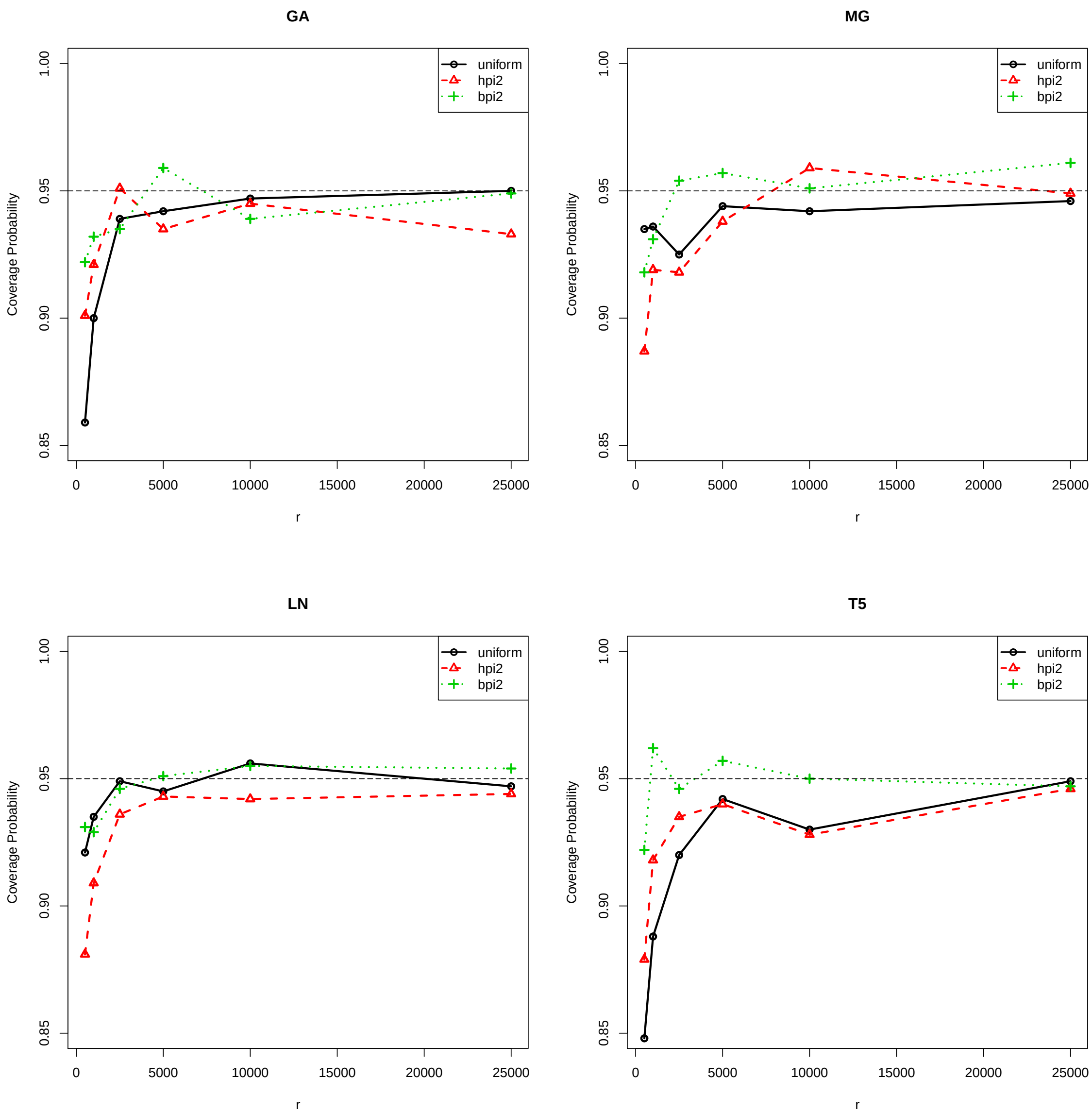


Figure 8: Same as Fig 7 except for the Negative Binomial model.

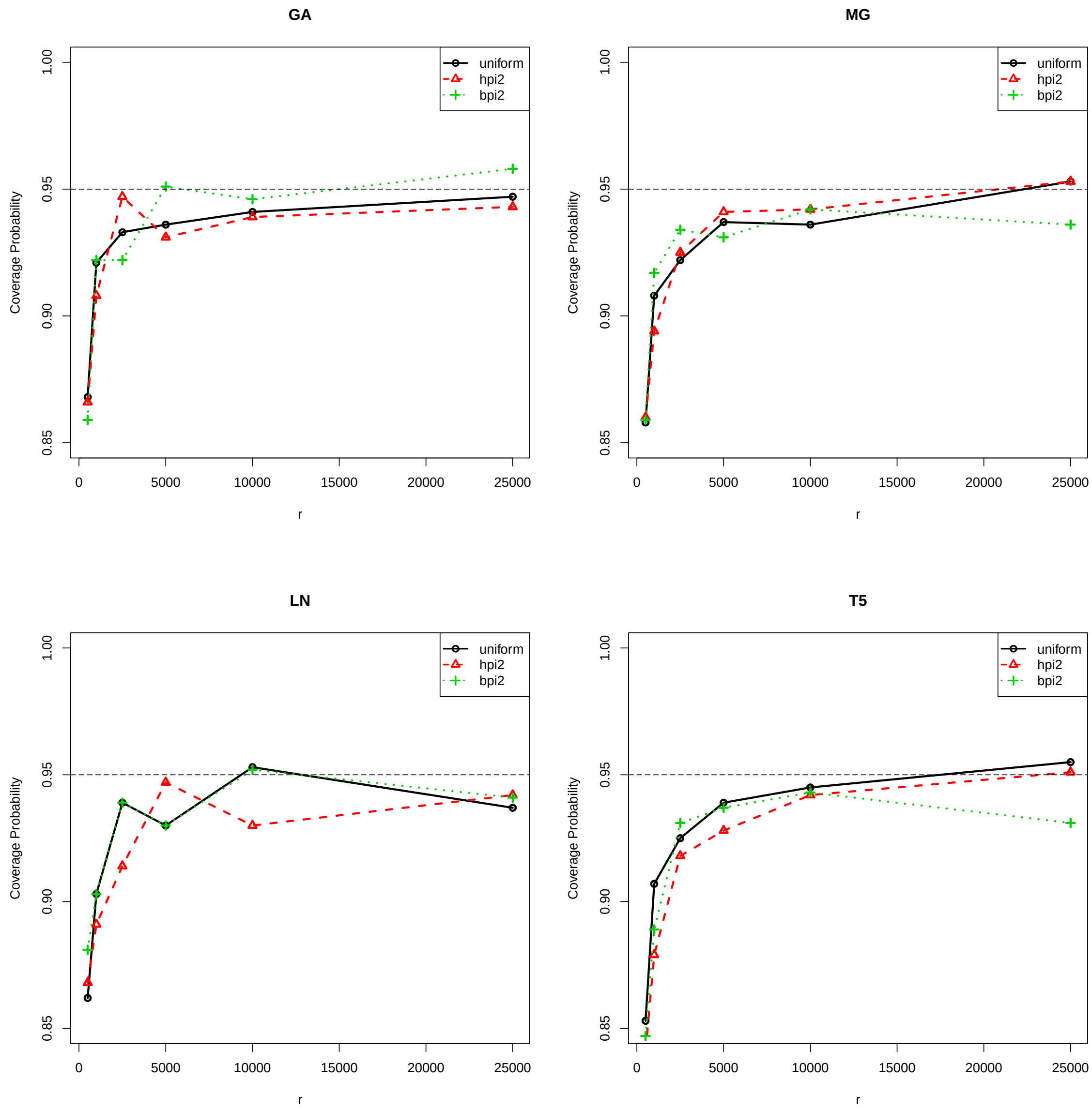


Table 1: Simulated MSE ratios of the subsampling estimator $\hat{\boldsymbol{\beta}}^*$ of the A-optimal sampling to the uniform in the Poisson model using the full-sample estimator $\hat{\boldsymbol{\beta}}$ for $n = 50,000$ and $p = 50$.

r	500	1000	2500	5000	10000	25000
r/n	1%	2%	5%	10%	20%	50%
			GA			
$\hat{\boldsymbol{\pi}}^{(2)}$	0.6533	0.5937	0.5627	0.5434	0.5343	0.5231
$\hat{\boldsymbol{\pi}}^{(1)}$	0.6554	0.6064	0.5613	0.5461	0.5322	0.5170
$\hat{\boldsymbol{\pi}}^{(0)}$	0.6494	0.6003	0.5672	0.5480	0.5346	0.5262
$\bar{\boldsymbol{\pi}}^{(2)}$	0.7715	0.7705	0.7898	0.7888	0.7972	0.7897
$\bar{\boldsymbol{\pi}}^{(1)}$	0.7665	0.7743	0.7960	0.7939	0.7975	0.8046
$\bar{\boldsymbol{\pi}}^{(0)}$	0.7592	0.7753	0.7794	0.8120	0.8020	0.7979
			MG			
$\hat{\boldsymbol{\pi}}^{(2)}$	0.3678	0.3642	0.3629	0.3558	0.3467	0.3524
$\hat{\boldsymbol{\pi}}^{(1)}$	0.3502	0.3502	0.3492	0.3478	0.3504	0.3588
$\hat{\boldsymbol{\pi}}^{(0)}$	0.3529	0.3536	0.3489	0.3465	0.3565	0.3609
$\bar{\boldsymbol{\pi}}^{(2)}$	0.4230	0.4521	0.4856	0.5137	0.5268	0.5451
$\bar{\boldsymbol{\pi}}^{(1)}$	0.4186	0.4567	0.4905	0.5166	0.5251	0.5608
$\bar{\boldsymbol{\pi}}^{(0)}$	0.4098	0.4436	0.4877	0.5193	0.5393	0.5466
			LN			
$\hat{\boldsymbol{\pi}}^{(2)}$	0.5328	0.5285	0.4992	0.4573	0.4823	0.4756
$\hat{\boldsymbol{\pi}}^{(1)}$	0.6002	0.5776	0.5177	0.4989	0.5560	0.5549
$\hat{\boldsymbol{\pi}}^{(0)}$	0.6267	0.5914	0.5418	0.5250	0.5200	0.5248
$\bar{\boldsymbol{\pi}}^{(2)}$	0.6602	0.6842	0.7031	0.7120	0.7010	0.7114
$\bar{\boldsymbol{\pi}}^{(1)}$	0.7049	0.7390	0.7586	0.7811	0.8152	0.8336
$\bar{\boldsymbol{\pi}}^{(0)}$	0.7348	0.7644	0.7840	0.7679	0.8163	0.7998
			T5			
$\hat{\boldsymbol{\pi}}^{(2)}$	0.3587	0.3137	0.2867	0.2714	0.2760	0.2810
$\hat{\boldsymbol{\pi}}^{(1)}$	0.3469	0.2987	0.2709	0.2608	0.2678	0.2784
$\hat{\boldsymbol{\pi}}^{(0)}$	0.3318	0.2872	0.2598	0.2578	0.2657	0.2822
$\bar{\boldsymbol{\pi}}^{(2)}$	0.4013	0.3695	0.3596	0.3636	0.3861	0.4229
$\bar{\boldsymbol{\pi}}^{(1)}$	0.3807	0.3527	0.3426	0.3562	0.3812	0.4207
$\bar{\boldsymbol{\pi}}^{(0)}$	0.3622	0.3351	0.3445	0.3629	0.3867	0.4240

Table 2: Same as Table 1 except for truncation 10%.

r	500	1000	2500	5000	10000	25000
r/n	1%	2%	5%	10%	20%	50%
			GA			
$\hat{\pi}^{(2)}$	0.6434	0.5718	0.5499	0.5325	0.5310	0.5159
$\hat{\pi}^{(1)}$	0.6271	0.5811	0.5450	0.5389	0.5265	0.5199
$\hat{\pi}^{(0)}$	0.6299	0.5823	0.5481	0.5410	0.5302	0.5163
$\bar{\pi}^{(2)}$	0.7730	0.7662	0.7898	0.7965	0.7980	0.7960
$\bar{\pi}^{(1)}$	0.7688	0.7658	0.8021	0.8032	0.7968	0.8079
$\bar{\pi}^{(0)}$	0.7701	0.7726	0.7866	0.8121	0.8122	0.7935
			MG			
$\hat{\pi}^{(2)}$	0.3671	0.3571	0.3534	0.3444	0.3483	0.3540
$\hat{\pi}^{(1)}$	0.3525	0.3527	0.3404	0.3464	0.3534	0.3633
$\hat{\pi}^{(0)}$	0.3488	0.3410	0.3441	0.3527	0.3524	0.3629
$\bar{\pi}^{(2)}$	0.4236	0.4483	0.4925	0.5070	0.5306	0.5486
$\bar{\pi}^{(1)}$	0.4201	0.4555	0.4938	0.5070	0.5340	0.5497
$\bar{\pi}^{(0)}$	0.4111	0.4473	0.4915	0.5202	0.5354	0.5510
			LN			
$\hat{\pi}^{(2)}$	0.5230	0.5275	0.4854	0.4466	0.4847	0.4764
$\hat{\pi}^{(1)}$	0.5865	0.5411	0.5404	0.4924	0.5453	0.5439
$\hat{\pi}^{(0)}$	0.5853	0.5853	0.5359	0.4973	0.5124	0.5395
$\bar{\pi}^{(2)}$	0.6571	0.6894	0.6773	0.6833	0.7002	0.7404
$\bar{\pi}^{(1)}$	0.6965	0.7325	0.7799	0.7791	0.8176	0.8318
$\bar{\pi}^{(0)}$	0.7126	0.7565	0.8055	0.7710	0.8076	0.8029
			T5			
$\hat{\pi}^{(2)}$	0.3538	0.3060	0.2815	0.2722	0.2753	0.2823
$\hat{\pi}^{(1)}$	0.3394	0.2900	0.2678	0.2595	0.2650	0.2817
$\hat{\pi}^{(0)}$	0.3233	0.2793	0.2604	0.2587	0.2659	0.2824
$\bar{\pi}^{(2)}$	0.4081	0.3721	0.3595	0.3680	0.3872	0.4241
$\bar{\pi}^{(1)}$	0.3844	0.3565	0.3451	0.3600	0.3812	0.4232
$\bar{\pi}^{(0)}$	0.3613	0.3356	0.3453	0.3667	0.3885	0.4258

Table 3: Same as Table 1 except for truncation 30%.

r	500	1000	2500	5000	10000	25000
r/n	1%	2%	5%	10%	20%	50%
			GA			
$\hat{\pi}^{(2)}$	0.6196	0.5769	0.5551	0.5372	0.5371	0.5317
$\hat{\pi}^{(1)}$	0.6198	0.5752	0.5480	0.5435	0.5381	0.5373
$\hat{\pi}^{(0)}$	0.6185	0.5723	0.5465	0.5486	0.5377	0.5345
$\bar{\pi}^{(2)}$	0.7816	0.7805	0.8033	0.8041	0.8077	0.8126
$\bar{\pi}^{(1)}$	0.7832	0.7811	0.8103	0.8055	0.8137	0.8168
$\bar{\pi}^{(0)}$	0.7774	0.7823	0.7997	0.8125	0.8140	0.8043
			MG			
$\hat{\pi}^{(2)}$	0.3667	0.3625	0.3568	0.3573	0.3536	0.3633
$\hat{\pi}^{(1)}$	0.3515	0.3556	0.3491	0.3544	0.3660	0.3674
$\hat{\pi}^{(0)}$	0.3502	0.3493	0.3488	0.3590	0.3515	0.3611
$\bar{\pi}^{(2)}$	0.4309	0.4629	0.4868	0.5226	0.5351	0.5524
$\bar{\pi}^{(1)}$	0.4250	0.4539	0.4985	0.5176	0.5354	0.5630
$\bar{\pi}^{(0)}$	0.4102	0.4484	0.4977	0.5193	0.5393	0.5617
			LN			
$\hat{\pi}^{(2)}$	0.5193	0.5118	0.4905	0.4791	0.4721	0.5021
$\hat{\pi}^{(1)}$	0.5619	0.5496	0.5325	0.5132	0.5637	0.5466
$\hat{\pi}^{(0)}$	0.5596	0.5675	0.5274	0.5120	0.5204	0.5371
$\bar{\pi}^{(2)}$	0.6654	0.6930	0.7116	0.7232	0.7366	0.7309
$\bar{\pi}^{(1)}$	0.6989	0.7329	0.7832	0.7546	0.8173	0.8252
$\bar{\pi}^{(0)}$	0.7181	0.7604	0.7909	0.7819	0.8316	0.8255
			T5			
$\hat{\pi}^{(2)}$	0.3608	0.3160	0.2893	0.2763	0.2826	0.2880
$\hat{\pi}^{(1)}$	0.3460	0.2966	0.2785	0.2700	0.2724	0.2864
$\hat{\pi}^{(0)}$	0.3295	0.2843	0.2629	0.2658	0.2763	0.2954
$\bar{\pi}^{(2)}$	0.4121	0.3778	0.3704	0.3645	0.3875	0.4239
$\bar{\pi}^{(1)}$	0.3882	0.3588	0.3493	0.3602	0.3860	0.4232
$\bar{\pi}^{(0)}$	0.3676	0.3374	0.3424	0.3639	0.3911	0.4198

Table 4: Same as Table 1 except for the Negative Binomial model.

r	500	1000	2500	5000	10000	25000
r/n	1%	2%	5%	10%	20%	50%
			GA			
$\hat{\pi}^{(2)}$	0.3390	0.3243	0.3328	0.3319	0.3310	0.3398
$\hat{\pi}^{(1)}$	0.3315	0.3259	0.3281	0.3359	0.3372	0.3358
$\hat{\pi}^{(0)}$	0.3374	0.3265	0.3408	0.3374	0.3333	0.3416
$\bar{\pi}^{(2)}$	0.9850	0.9747	0.9765	0.9699	0.9668	0.9738
$\bar{\pi}^{(1)}$	0.9881	0.9752	0.9775	0.9747	0.9695	0.9845
$\bar{\pi}^{(0)}$	0.9992	0.9739	0.9913	0.9955	0.9978	0.9757
			MG			
$\hat{\pi}^{(2)}$	0.2843	0.2863	0.2924	0.2974	0.3078	0.3132
$\hat{\pi}^{(1)}$	0.2863	0.2819	0.2908	0.3030	0.3040	0.3107
$\hat{\pi}^{(0)}$	0.2854	0.2831	0.2922	0.3004	0.3076	0.3129
$\bar{\pi}^{(2)}$	0.9295	0.9020	0.9000	0.8748	0.9118	0.9040
$\bar{\pi}^{(1)}$	0.9164	0.8945	0.9006	0.8936	0.9203	0.9243
$\bar{\pi}^{(0)}$	0.9347	0.9163	0.9142	0.8970	0.9152	0.9229
			LN			
$\hat{\pi}^{(2)}$	0.3208	0.2963	0.2923	0.3214	0.3148	0.3229
$\hat{\pi}^{(1)}$	0.3447	0.3214	0.3389	0.3364	0.3584	0.3603
$\hat{\pi}^{(0)}$	0.3409	0.3361	0.3454	0.3474	0.3590	0.3554
$\bar{\pi}^{(2)}$	0.8698	0.8666	0.8634	0.8762	0.9167	0.9062
$\bar{\pi}^{(1)}$	0.9364	0.9482	0.9942	0.9643	0.9789	0.9733
$\bar{\pi}^{(0)}$	0.9197	0.9289	0.9370	0.9564	0.9849	0.9673
			T5			
$\hat{\pi}^{(2)}$	0.3013	0.2923	0.2844	0.2955	0.2986	0.3053
$\hat{\pi}^{(1)}$	0.2979	0.2933	0.2863	0.2956	0.2983	0.3027
$\hat{\pi}^{(0)}$	0.3034	0.2898	0.2924	0.2944	0.2998	0.3014
$\bar{\pi}^{(2)}$	0.9115	0.8764	0.8493	0.8565	0.8599	0.8543
$\bar{\pi}^{(1)}$	0.9087	0.8787	0.8516	0.8658	0.8545	0.8632
$\bar{\pi}^{(0)}$	0.9107	0.8861	0.8461	0.8546	0.8730	0.8752

Table 5: Same as Table 4 except for truncation 10%.

r	500	1000	2500	5000	10000	25000
r/n	1%	2%	5%	10%	20%	50%
			GA			
$\hat{\pi}^{(2)}$	0.3158	0.3146	0.3273	0.3301	0.3375	0.3390
$\hat{\pi}^{(1)}$	0.3158	0.3184	0.3284	0.3253	0.3370	0.3366
$\hat{\pi}^{(0)}$	0.3171	0.3162	0.3269	0.3308	0.3362	0.3391
$\bar{\pi}^{(2)}$	0.9836	0.9777	0.9828	0.9807	0.9761	0.9685
$\bar{\pi}^{(1)}$	0.9797	0.9771	0.9901	0.9670	0.9732	0.9783
$\bar{\pi}^{(0)}$	0.9693	0.9804	0.9931	0.9763	0.9792	0.9720
			MG			
$\hat{\pi}^{(2)}$	0.2793	0.2801	0.2901	0.2987	0.3014	0.3108
$\hat{\pi}^{(1)}$	0.2756	0.2722	0.2888	0.3009	0.3048	0.3099
$\hat{\pi}^{(0)}$	0.2793	0.2762	0.2959	0.2989	0.3078	0.3116
$\bar{\pi}^{(2)}$	0.9420	0.9153	0.9214	0.9208	0.8974	0.9136
$\bar{\pi}^{(1)}$	0.9524	0.9160	0.9236	0.9062	0.8968	0.9199
$\bar{\pi}^{(0)}$	0.9404	0.9213	0.9039	0.9301	0.9096	0.9147
			LN			
$\hat{\pi}^{(2)}$	0.2936	0.2887	0.2768	0.3003	0.3024	0.3175
$\hat{\pi}^{(1)}$	0.3125	0.3169	0.3067	0.3230	0.3348	0.3719
$\hat{\pi}^{(0)}$	0.3233	0.3062	0.3069	0.3233	0.3294	0.3652
$\bar{\pi}^{(2)}$	0.8520	0.8418	0.8104	0.8698	0.8743	0.8878
$\bar{\pi}^{(1)}$	0.8721	0.9179	0.8642	0.9182	0.9409	0.9457
$\bar{\pi}^{(0)}$	0.9088	0.9586	0.8802	0.8937	0.9499	0.9804
			T5			
$\hat{\pi}^{(2)}$	0.2855	0.2843	0.2843	0.2881	0.2902	0.3015
$\hat{\pi}^{(1)}$	0.2871	0.2817	0.2812	0.2910	0.2969	0.3014
$\hat{\pi}^{(0)}$	0.2875	0.2819	0.2842	0.2903	0.2991	0.2960
$\bar{\pi}^{(2)}$	0.8808	0.8615	0.8441	0.8464	0.8579	0.8484
$\bar{\pi}^{(1)}$	0.8945	0.8723	0.8583	0.8475	0.8497	0.8476
$\bar{\pi}^{(0)}$	0.8965	0.8792	0.8621	0.8601	0.8516	0.8470

Table 6: Same as Table 4 except for truncation 30%.

r	500	1000	2500	5000	10000	25000
r/n	1%	2%	5%	10%	20%	50%
			GA			
$\hat{\pi}^{(2)}$	0.3163	0.3154	0.3307	0.3333	0.3416	0.3471
$\hat{\pi}^{(1)}$	0.3091	0.3200	0.3286	0.3334	0.3417	0.3435
$\hat{\pi}^{(0)}$	0.3163	0.3199	0.3363	0.3349	0.3400	0.3476
$\bar{\pi}^{(2)}$	0.9869	0.9854	0.9928	0.9835	0.9831	0.9716
$\bar{\pi}^{(1)}$	0.9797	0.9859	0.9910	0.9656	0.9860	0.9828
$\bar{\pi}^{(0)}$	0.9671	0.9795	0.9934	0.9739	0.9882	0.9734
			MG			
$\hat{\pi}^{(2)}$	0.2735	0.2780	0.2944	0.3023	0.3077	0.3155
$\hat{\pi}^{(1)}$	0.2715	0.2762	0.2930	0.3069	0.3116	0.3143
$\hat{\pi}^{(0)}$	0.2796	0.2809	0.2962	0.3068	0.3150	0.3187
$\bar{\pi}^{(2)}$	0.9551	0.9141	0.9206	0.9370	0.9004	0.9148
$\bar{\pi}^{(1)}$	0.9483	0.9256	0.9297	0.9127	0.9088	0.9279
$\bar{\pi}^{(0)}$	0.9340	0.9295	0.9061	0.9305	0.9129	0.9191
			LN			
$\hat{\pi}^{(2)}$	0.2909	0.2874	0.2925	0.3050	0.2907	0.3126
$\hat{\pi}^{(1)}$	0.3255	0.3129	0.3126	0.3418	0.3258	0.3435
$\hat{\pi}^{(0)}$	0.3119	0.3235	0.3249	0.3390	0.3134	0.3446
$\bar{\pi}^{(2)}$	0.8524	0.8349	0.8412	0.8808	0.8313	0.8860
$\bar{\pi}^{(1)}$	0.8938	0.8668	0.9118	0.9429	0.8637	0.9391
$\bar{\pi}^{(0)}$	0.8918	0.8835	0.9237	0.9518	0.9301	0.9241
			T5			
$\hat{\pi}^{(2)}$	0.2921	0.2842	0.2888	0.2932	0.2981	0.3083
$\hat{\pi}^{(1)}$	0.2880	0.2847	0.2876	0.2957	0.3039	0.3047
$\hat{\pi}^{(0)}$	0.2867	0.2885	0.2919	0.2911	0.2998	0.3048
$\bar{\pi}^{(2)}$	0.8796	0.8935	0.8612	0.8459	0.8582	0.8555
$\bar{\pi}^{(1)}$	0.8767	0.8819	0.8668	0.8484	0.8623	0.8532
$\bar{\pi}^{(0)}$	0.8964	0.8898	0.8797	0.8615	0.8484	0.8537

Table 7: Same as Table 1 except for using the Scoring Algorithm (instead of the full-sample $\hat{\boldsymbol{\beta}}$) with presample size $r_0 = 500$.

r	500	1000	2500	5000	10000	25000
r/n	1%	2%	5%	10%	20%	50%
			GA			
$\hat{\boldsymbol{\pi}}^{(2)}$	0.7778	0.7375	0.7749	0.8050	0.8276	0.8499
$\hat{\boldsymbol{\pi}}^{(1)}$	0.7794	0.7594	0.7781	0.7898	0.8259	0.8778
$\hat{\boldsymbol{\pi}}^{(0)}$	0.7792	0.7657	0.7750	0.8096	0.8413	0.8725
$\bar{\boldsymbol{\pi}}^{(2)}$	0.7805	0.7879	0.8036	0.8237	0.8300	0.8205
$\bar{\boldsymbol{\pi}}^{(1)}$	0.7930	0.7888	0.8188	0.8341	0.8271	0.8174
$\bar{\boldsymbol{\pi}}^{(0)}$	0.7911	0.7967	0.8293	0.8419	0.8494	0.8416
			MG			
$\hat{\boldsymbol{\pi}}^{(2)}$	0.4192	0.4869	0.5671	0.6089	0.7003	0.7533
$\hat{\boldsymbol{\pi}}^{(1)}$	0.4339	0.5021	0.5856	0.6567	0.7313	0.7869
$\hat{\boldsymbol{\pi}}^{(0)}$	0.4486	0.5219	0.5941	0.6723	0.7247	0.7884
$\bar{\boldsymbol{\pi}}^{(2)}$	0.4270	0.4712	0.4905	0.5279	0.5555	0.5557
$\bar{\boldsymbol{\pi}}^{(1)}$	0.4195	0.4579	0.5144	0.5157	0.5618	0.5620
$\bar{\boldsymbol{\pi}}^{(0)}$	0.4254	0.4603	0.4854	0.5371	0.5735	0.5805
			LN			
$\hat{\boldsymbol{\pi}}^{(2)}$	0.6271	0.6467	0.6639	0.6623	0.7056	0.7639
$\hat{\boldsymbol{\pi}}^{(1)}$	0.6990	0.7057	0.6935	0.7226	0.8034	0.8218
$\hat{\boldsymbol{\pi}}^{(0)}$	0.7114	0.7384	0.7262	0.7301	0.8335	0.8643
$\bar{\boldsymbol{\pi}}^{(2)}$	0.6606	0.6884	0.7185	0.7238	0.7160	0.7500
$\bar{\boldsymbol{\pi}}^{(1)}$	0.6960	0.7362	0.7549	0.7833	0.8286	0.8412
$\bar{\boldsymbol{\pi}}^{(0)}$	0.7329	0.7824	0.8193	0.7992	0.8546	0.8145
			T5			
$\hat{\boldsymbol{\pi}}^{(2)}$	0.3184	0.3077	0.2828	0.2969	0.3139	0.3291
$\hat{\boldsymbol{\pi}}^{(1)}$	0.3079	0.2933	0.2964	0.2957	0.3111	0.3295
$\hat{\boldsymbol{\pi}}^{(0)}$	0.3260	0.3087	0.3022	0.3084	0.3240	0.3419
$\bar{\boldsymbol{\pi}}^{(2)}$	0.3956	0.3808	0.3626	0.3719	0.3927	0.4156
$\bar{\boldsymbol{\pi}}^{(1)}$	0.3744	0.3483	0.3500	0.3596	0.3853	0.4209
$\bar{\boldsymbol{\pi}}^{(0)}$	0.3419	0.3425	0.3521	0.3628	0.3967	0.4285

Table 8: Same as Table 7 except for the Negative Binomial model.

r	500	1000	2500	5000	10000	25000
r/n	1%	2%	5%	10%	20%	50%
			GA			
$\hat{\pi}^{(2)}$	0.3814	0.3811	0.3822	0.3847	0.3840	0.3920
$\hat{\pi}^{(1)}$	0.3837	0.3790	0.3835	0.3858	0.3867	0.4050
$\hat{\pi}^{(0)}$	0.3788	0.3841	0.3851	0.3821	0.3868	0.3954
$\bar{\pi}^{(2)}$	1.0045	0.9018	0.9891	0.9718	0.9738	0.9896
$\bar{\pi}^{(1)}$	0.9895	0.9757	0.9905	0.9849	0.9701	0.9860
$\bar{\pi}^{(0)}$	0.9831	0.9543	0.9872	0.9744	0.9925	0.9855
			MG			
$\hat{\pi}^{(2)}$	0.3140	0.3386	0.3478	0.3578	0.3852	0.3800
$\hat{\pi}^{(1)}$	0.3232	0.3385	0.3438	0.3672	0.3859	0.3866
$\hat{\pi}^{(0)}$	0.3300	0.3405	0.3480	0.3670	0.3803	0.3801
$\bar{\pi}^{(2)}$	0.9098	0.9207	0.8999	0.9189	0.9346	0.8895
$\bar{\pi}^{(1)}$	0.9286	0.9233	0.9198	0.9253	0.9341	0.9117
$\bar{\pi}^{(0)}$	0.9521	0.9209	0.9021	0.9141	0.9454	0.9161
			LN			
$\hat{\pi}^{(2)}$	0.3759	0.3577	0.3380	0.3625	0.3892	0.3796
$\hat{\pi}^{(1)}$	0.4049	0.3750	0.3696	0.3977	0.4197	0.4378
$\hat{\pi}^{(0)}$	0.3976	0.3793	0.3573	0.3858	0.4499	0.4148
$\bar{\pi}^{(2)}$	0.8391	0.8651	0.8383	0.8846	0.9278	0.9738
$\bar{\pi}^{(1)}$	0.9403	0.9732	0.8511	0.9292	0.9367	0.9631
$\bar{\pi}^{(0)}$	0.9426	0.9851	0.9166	0.9415	0.9132	0.9970
			T5			
$\hat{\pi}^{(2)}$	0.3473	0.3404	0.3480	0.3473	0.3576	0.3747
$\hat{\pi}^{(1)}$	0.3521	0.3383	0.3498	0.3526	0.3601	0.3620
$\hat{\pi}^{(0)}$	0.3462	0.3426	0.3473	0.3573	0.3622	0.3672
$\bar{\pi}^{(2)}$	0.8952	0.8512	0.8679	0.8400	0.8387	0.8497
$\bar{\pi}^{(1)}$	0.8704	0.8551	0.8690	0.8528	0.8337	0.8557
$\bar{\pi}^{(0)}$	0.9097	0.8591	0.8697	0.8583	0.8465	0.8518

Table 9: The CPU times in seconds for computing $\hat{\boldsymbol{\beta}}_r^*$ using $\hat{\pi}^{(2)}$ and $\bar{\pi}^{(2)}$ in the Poisson model with GA data-generation using the Scoring Algorithm for pre-subsample size $r_0 = 500$, $n = 50,000$ and $p = 50$.

r	500	1000	1500	2000	2500	5000
r/n	1%	2%	3%	4%	5%	10%
$\hat{\pi}^{(2)}$	4.191	4.205	4.226	4.241	4.567	4.632
$\bar{\pi}^{(2)}$	2.313	2.334	2.356	2.395	3.025	3.564
Full data CPU time 5.872 seconds						

Table 10: The CPU times in seconds using Newton's Algorithm for the full-sample sizes in the Poisson model with GA data-generation for pre-sample size $r_0 = 500$ and $r = 2,000$.

n	10^4	10^5	10^6	0.5×10^7
$\hat{\pi}^{(2)}$	0.70	4.67	26.30	98.06
$\bar{\pi}^{(2)}$	0.64	3.50	15.22	49.22
Full	0.76	6.59	58.26	299.18

Table 11: The average of the iterations in Newton's algorithm using the Poisson model with GA-data generation for $r_0 = 500$ and various r. The iterations for the full data are 8.4.

	$\hat{\boldsymbol{\pi}}^{(2)}$		$\bar{\boldsymbol{\pi}}^{(2)}$		Uniform
r	Step1	Step2	Step1	Step2	
500	8.89	8.77	8.67	8.49	8.40
1000	8.75	8.56	8.56	8.23	8.80
1500	8.56	8.32	8.59	8.39	8.54
2000	8.55	8.01	8.58	8.53	8.34
2500	8.60	8.91	8.62	8.85	8.27

5 The A-optimal Subsampling for the Analysis of the Blog Feedback Data

In this section, we apply the Subsampling approach to analyzing the *Blog Feedback* data using the Poisson, the Quasipoisson and the Zero-Inflated regression models. The sampling distributions were calculated using the Zero-Inflated model in (2.5) with the estimates in (5.1) and the discussion therein, and the Poisson and the Quasipoisson models in (3.11).

The data set is available from the UCI machine learning repository (URL: https://archive.ics.uci.edu/), and was collected and processed from raw html of the blog posts. The goal is to *predict the number of comments in the upcoming 24 hours relative to the base time*. The base time was chosen from the past, and the blog posts selected were published within 72 hours before the base time. The features were recorded at the base time based on the selected blog posts.

There are $52,397$ observations in the training data, and $7,624$ observations in the test data. We used the training data to build the model, and the test data to calculate the prediction errors. There are 23 features, see Table 12.

The Poisson model is not appropriate for this data because of the observed overdispersion and inflated number of zeros. The Quasipoisson model has the same parameter estimates as the Poisson model and does not accommodate zero-inflation, it is thus not a good choice either. The Zero-inflated Poisson model allows inflated zeros and is an appropriate choice.

As 64.05% of the values in the response variable are 0, we shall consider fitting the zero-inflated Poisson regression model in (2.5) for the data. The estimating equation of the model contains the parameter $0 \leq \rho \leq 1$, which accounts for the amount of positive structural zeros beyond the sampling zeros explained by the Poisson distribution f_{poi}. In the literature, ρ can be modeled as a function of the predictor variables, for example, via the logistic link. Here for simplifying the estimating process, we shall estimate ρ first. Specifically, based on the interpretation of ρ and noting that 64.05% is the proportion of zeros in the response variable while $\exp(\mu)$ is the probability of taking zero value in the Poisson distribution, we estimate ρ by

$$\hat{\rho} = 0.6405 - \exp(-\hat{\mu}), \tag{5.1}$$

where $\hat{\mu}$ is an estimator of μ. As Y follows the Zero-Inflated model (2.5), we have

$$\mathrm{P}(Y = 0) = \rho + (1 - \rho)\exp(-\mu).$$

On the other hand, $\mathrm{E}(Y) = (1 - \rho)\mu$. Thus $\mu = \mathrm{E}(Y)/(1 - \rho)$ and we get

$$p_0 = \mathrm{P}(Y = 0) = \rho + (1 - \rho)\exp(-\mathrm{E}(Y)/(1 - \rho)).$$

The empirical estimate of p_0 is

$$\hat{p}_0 = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}[Y_i = 0] = \rho + (1 - \rho)\exp(-\bar{Y}/(1 - \rho)).$$

As $\bar{y} = 6.765$ and $\hat{p}_0 = 0.6405$, we get $\hat{\rho} \approx \hat{p}_0 = 0.6405$. Alternatively, we can use (5.1) to get $\hat{\rho}$ by plugging in $\hat{\mu} = 214.9628$, yielding the same value.

To compare the Poisson model and the Quasipoisson model with the Zero-Inflated Poisson model, we report the full-sample estimates, the standard errors, the P-values for the three models in Table 13. Many parameters in the Quasipoisson model were not significant, while these parameters in the Zero-Inflated Poisson model were significant.

In Table 14, observe that the standard errors of the uniform method were bigger than those of $\hat{\boldsymbol{\pi}}^{(2)}$ method, the averageds of the parameter estimates using the $\hat{\boldsymbol{\pi}}^{(2)}$ sampling were closer to the full-sample estimates than those using the uniform.

For the uniform subsampling, the comparison between the theoretical and the empirical standard errors showed large differences. This means the empirical performance of the uniform subsampling did not reach the theoretical values in the presence of inflated zeros for $r = 5000$. The theoretical and the empirical standard errors of the $\hat{\boldsymbol{\pi}}^{(2)}$ subsampling indicated that the empirical performance is consistent with the theoretical results in presence of infalated zeros.

The comparison between the uniform and $\hat{\boldsymbol{\pi}}^{(2)}$ subsampling suggested that for many variables the P-value using the $\hat{\boldsymbol{\pi}}^{(2)}$ subsampling were significant while that of the uniform subsampling were not. For example, the effects of Tc, Cl24, Cf24, Tt, Tl24, Tt1t2, Mbt, Fbt, Sbt, Mpb, Tpb, THpb, Fpb, Spb, and Ppage were detected by using the $\hat{\boldsymbol{\pi}}^{(2)}$ subsampling but were not detected by using the uniform. This means the proposed approach reduced the standard error hence increased the power of the test for the regression coefficients.

Tables 15–16 are the ratios of the lengths of the confidence intervals and the coverage probabilities. In Table 15, all the values are smaller than 1, indicating that the lengths of 95% confidence intervals constructed by using $\hat{\boldsymbol{\pi}}^{(2)}$ subsampling were significantly smaller than those of the uniform.

Table 17 shows the MSE ratios of $\hat{\boldsymbol{\pi}}^{(2)}$ subsampling to the uniform. The values were smaller than 0.1, indicating that the MSE of our optimal Subsampling approach was less than 10% percent of the MSE of using the uniform subsampling.

Table 18 reports the averages of the sum of squared predicted errors, and Fig. 9 graphically represents Table 18, one observes that when r was small, the uniform subsampling produced very large prediction errors. The prediction errors produced by using the $\hat{\boldsymbol{\pi}}^{(2)}$ subsampling were significantly smaller than those by using the uniform.

Table 12: 23 Features In the Blog Feedback Data

Tc	Total number of comments before basetime
Cl24	Number of comments in the 24 hours right before the basetime
Ct1t2	Number of comments in the time period between $T1$ and $T2$, where $T1$ denotes the date time 48 hours before basetime, $T2$ denotes the date time 24 hours before basetime,
Cf24	Number of comments in 24 hours immediately after publication of the post but before basetime
Tt	Total number of trackbacks before basetime,
Tl24	Number of trackbacks in the last 24 hours before the basettime
Tt1t2	Number of trackbacks between T1 and T2, where T1 is the time point 48 hours before basetime and T2 the time point 24 hours before basetime
Tf24	Number of trackbacks in 24 hours immediately after publication of the post but before basetime
Ltime	Length of time between the publication of the blog post and basetime
Lbp	Length of the blog post
Mbt, Tbt, Wbt THbt, Fbt, Sbt	Indicators (0 or 1) for whether Monday to whether Saturday of the basetime,
Mpb, Tpb, Wpb THpb, Fpb, Spb	Indicators (0 or 1) for whether Monday to whether Saturday of the blog publication date
Ppage	Number of parent pages.

Table 13: The estimates, standard errors, and P-values in Poisson, Quasipoisson, and Zero-Inflated Poisson models using the full-sample $n = 52,397$.

	Poisson	SE	P-value	Quasipoisson	SE	P-value	ZIPoisson	SE	P-value
Intcpt	2.70536	.01058	$< .0001$	2.70536	.08167	$< .0001$	3.42978	.01085	$< .0001$
Tc	.00371	.00004	$< .0001$	.00371	.00030	$< .0001$	.00312	.00004	$< .0001$
Cl24	.00282	.00004	$< .0001$	.00282	.00030	$< .0001$	.00276	.00004	$< .0001$
Ct1t2	.00013	.00005	.00373	.00013	.00036	.70717	.00025	.00005	$< .0001$
Cf24	-.00236	.00002	$< .0001$	-.00236	.00019	$< .0001$	-.00254	.00003	$< .0001$
Tt	.18007	.00482	$< .0001$	.18007	.03719	$< .0001$	.15279	.00471	$< .0001$
Tl24	-.09276	.00280	$< .0001$	-.09276	.02165	.00002	-.09377	.00267	$< .0001$
Tt1t2	-.03809	.00313	$< .0001$	-.03809	.02412	.11438	-.04378	.00298	$< .0001$
Tf24	-.06000	.00456	$< .0001$	-.06000	.03520	.08830	-.03660	.00445	$< .0001$
Ltime	-.06277	.00014	$< .0001$	-.06277	.00107	$< .0001$	-.05235	.00015	$< .0001$
Lbp	.00005	$< .0001$	$< .0001$	.00005	.00001	$< .0001$	.00004	.00001	$< .0001$
Mbt	.19249	.00912	$< .0001$	.19249	.07040	.00626	.09339	.00933	$< .0001$
Tbt	.07939	.01072	$< .0001$	.07939	.08276	.33744	-.06151	.01122	$< .0001$
Wbt	.02238	.01104	.04267	.02238	.08523	.79289	-.13030	.01155	$< .0001$
THbt	.05547	.01067	$< .0001$	.05547	.08238	.50077	-.09195	.01108	$< .0001$
Fbt	-.24868	.00977	$< .0001$	-.24868	.07542	.00098	-.31279	.01002	$< .0001$
Sbt	-.23916	.00794	$< .0001$	-.23916	.06128	.00010	-.22643	.00805	$< .0001$
Mpb	.18675	.00992	$< .0001$	.18675	.07658	.01474	.15946	.01051	$< .0001$
Tpb	.23210	.01107	$< .0001$	.23210	.08547	.00662	.22193	.01169	$< .0001$
Wpb	.05575	.01158	$< .0001$	.05575	.08935	.53271	.08395	.01204	$< .0001$
THpb	.36164	.01134	$< .0001$	.36164	.08755	.00004	.29686	.01174	$< .0001$
Fpb	.47488	.01037	$< .0001$	.47488	.08004	$< .0001$	.33577	.01060	$< .0001$
Spb	.19624	.00984	$< .0001$	.19624	.07599	.00982	.09328	.01011	$< .0001$
Ppage	-.17265	.00389	$< .0001$	-.17265	.03005	$< .0001$	-.11498	.00363	$< .0001$

Table 14: The averages of the estimates, theoretical standard errors (Tse), empirical standard errors (Ese), and P-values based on 1000 subsamples in the Zero-Inflated Poisson model with $r_0 = 2500$ and $r = 5000$.

	unif				$\hat{\boldsymbol{\pi}}^{(2)}$			
	Estimate	Tse	Ese	P-value	Estimate	Tse	Ese	P-value
Intercept	3.31604	.60943	.56943	$< .0001$	3.39144	.08391	.07782	$< .0001$
Tc	.00499	.00463	.00271	.28070	.00318	.00036	.00029	$< .0001$
Cl24	.00298	.00246	.00172	.22524	.00270	.00031	.00029	$< .0001$
Ct1t2	-.00006	.00275	.00212	.98193	.00024	.00037	.00034	.51017
Cf24	-.00407	.00332	.00266	.22057	-.00252	.00026	.00019	$< .0001$
Tt	.13274	.60876	.32564	.82739	.15511	.04384	.03117	.00040
Tl24	-.08212	.12111	.12792	.49776	-.09500	.01955	.02092	$< .0001$
Tt1t2	-.04429	.13443	.14660	.74182	-.04496	.02124	.02190	.03431
Tf24	-.02871	.62134	.32695	.96314	-.03759	.04335	.02892	.38585
Ltime	-.05948	.00787	.00744	$< .0001$	-.05443	.00168	.00164	$< .0001$
Lbp	.00003	.00001	.00001	.02618	.00004	.00001	.00001	$< .0001$
Mbt	.15348	.49478	.47033	.75641	.13700	.06758	.06715	.04264
Tbt	.01812	1.01089	.59248	.98570	-.07086	.09689	.10023	.46461
Wbt	-.15888	.94287	.61652	.86618	-.15383	.10749	.10257	.15239
THbt	-.11398	.85801	.59002	.89431	-.08562	.10579	.10892	.41830
Fbt	-.25691	.72860	.53337	.72438	-.25842	.09592	.11123	.00706
Sbt	-.25243	.62211	.43792	.68491	-.23805	.08079	.08066	.00321
Mpb	.18671	.73179	.49846	.79861	.22473	.07984	.10999	.00488
Tpb	.36845	.76743	.57054	.63115	.30397	.10215	.11611	.00292
Wpb	.23372	.71303	.59162	.74307	.12763	.10816	.11870	.23797
THpb	.33222	.64228	.61171	.60499	.29644	.10121	.12326	.00340
Fpb	.49398	.60383	.56036	.41331	.40595	.08629	.09056	$< .0001$
Spb	.24244	.57235	.50639	.67186	.14735	.07293	.07362	.04333
Ppage	-.13385	.10897	.12224	.21934	-.11631	.03527	.03816	.00097

Table 15: The length ratios of the 95% confidence intervals of the $\hat{\boldsymbol{\pi}}^{(2)}$ subsampling to the uniform in the Zero-Inflated Poisson model with the pre-subsample size $r_0 = 2500$.

r	1000	2500	5000	10000	25000	50000
Intercept	.1950	.1368	.1441	.1499	.1263	.1315
Tc	.0901	.0976	.1069	.0953	.0930	.1012
Cl24	.1452	.1532	.1792	.1583	.1571	.1568
Ct1t2	.1440	.1481	.1676	.1448	.1491	.1469
Cf24	.0599	.0633	.0720	.0767	.0736	.0837
Tt	.0848	.0761	.0915	.0863	.0792	.0810
Tl24	.1048	.1152	.1579	.1522	.1641	.1751
Tt1t2	.1055	.1129	.1522	.1498	.1555	.1645
Tf24	.0932	.0776	.0890	.0797	.0734	.0786
Ltime	.2348	.2432	.2191	.2124	.2308	.2093
Lbp	.3005	.2605	.2890	.3476	.2928	.2687
Mbt	.2015	.1284	.1478	.1625	.1280	.1324
Tbt	.1950	.1338	.1526	.1611	.1092	.1218
Wbt	.1829	.1543	.1669	.1694	.1312	.1289
THbt	.2104	.1747	.1678	.1635	.1480	.1399
Fbt	.2024	.1512	.1632	.1571	.1486	.1556
Sbt	.2066	.1591	.1737	.1632	.1438	.1602
Mpb	.1602	.1746	.1666	.1742	.1548	.1285
Tpb	.1756	.1622	.1721	.1910	.1795	.1407
Wpb	.1782	.1923	.1725	.1979	.1810	.1415
THpb	.1922	.1623	.1647	.1770	.1549	.1568
Fpb	.1597	.1447	.1662	.1782	.1426	.1423
Spb	.1664	.1336	.1333	.1430	.1242	.1210
Ppage	.2388	.2534	.3325	.2951	.2876	.3433

Table 16: The simulated percentages of the 95% confidence intervals which caught the full sample MLE in the Zero-Inflated Poisson model with the pre-subsample size $r_0 = 2500$.

r	1000	2500	5000	10000	25000	50000
Intercept	.9989	.9989	.9919	.9955	.9924	.9917
Tc	.9905	.9956	.9902	.9979	.9915	.9999
Cl24	.9998	.9916	.9947	.9941	.9995	.9938
Ct1t2	.9965	.9986	.9989	.9940	.9923	.9928
Cf24	.9981	.9973	.9977	.9977	.9971	.9958
Tt	.9959	.9942	.9933	.9991	.9941	.9922
Tl24	.9901	.9936	.9951	.9947	.9998	.9979
Tt1t2	.9916	.9998	.9950	.9916	.9928	.9961
Tf24	.9994	.9976	.9949	.9986	.9920	.9918
Ltime	.9907	.9944	.9922	.9944	.9917	.9984
Lbp	.9903	.9998	.9997	.9936	.9934	.9948
Mbt	.9940	.9903	.9971	.9932	.9908	.9948
Tbt	.9992	.9998	.9972	.9922	.9989	.9970
Wbt	.9952	.9916	.9938	.9927	.9926	.9979
THbt	.9931	.9918	.9905	.9914	.9947	.9930
Fbt	.9983	.9987	.9949	.9962	.9934	.9955
Sbt	.9990	.9978	.9932	.9949	.9914	.9995
Mpb	.9978	.9986	.9936	1.0000	.9999	.9911
Tpb	.9918	.9961	.9944	.9987	.9906	.9990
Wpb	.9950	.9900	.9919	.9922	.9974	.9951
THpb	.9917	.9958	.9945	.9945	.9963	.9984
Fpb	.9979	.9986	.9932	.9966	.9957	.9998
Spb	.9988	.9929	.9925	.9994	.9996	.9905
Ppage	.9953	.9959	.9917	.9907	.9980	.9958

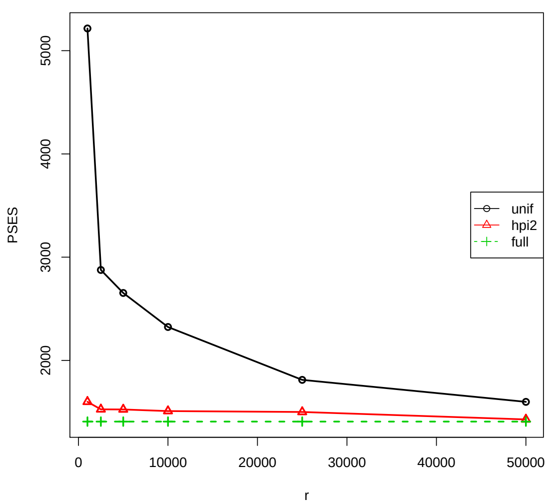
Table 17: The MSE ratios of the $\hat{\boldsymbol{\pi}}^{(2)}$ subsampling to the uniform in the Zero-Inflated Poisson regression model, with the pre-subsample size $r_0 = 2500$.

r	1000	2500	5000	10000	25000	50000
$\hat{\boldsymbol{\pi}}^{(2)}$	0.0287	0.0360	0.0373	0.0396	0.05796	0.0823

Table 18: The averages of the sum of squared predicted errors in the Zero-Inflated Poisson regression model, with pre-subsample size $r_0 = 2500$, the sum of the squared prediction error is $1,407.4712$ in the full sample.

r	1000	2500	5000	10000	25000	50000
uniform	5215.3313	2876.1691	2653.2441	2323.7320	1811.6740	1598.1760
$\hat{\boldsymbol{\pi}}^{(2)}$	1599.7506	1525.9297	1524.9536	1509.2128	1500.5681	1428.4280

Figure 9: The plot of the averaged predicted sum of squared errors in the Zero-Inflated Poisson model with $r_0 = 2500$.



References

[1] AVRON, H., MAYMOUNKOV, P. and TOLEDO, S. (2010). Blendenpik: Supercharging LAPACK's least-squares solver. *SIAM Journal on Scientific Computing*, **32**: 1217-1236.

[2] CAMERON, C. AND TRIVEDI, P. (1998). *Regression analysis of count data*. Cambridge University Press, UK.

[3] DOBSON, A. AND BARNETT, A. (2002). *An Introduction to Generalized Linear Models*. CRC Press, Boca Raton, FL.

[4] Drineas P., Kannan R. and Mahoney M.W. (2006a). Fast Monte Carlo algorithms for matrices I: Approximating matrix multiplication. *SIAM Journal on Computing*, **36**: 132-157.

[5] Drineas P., Mahoney M.W. and Muthukrishnan S. (2006b). Sampling algorithms for l2 regression and applications. *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms*, Pages: 1127-1136.

[6] Drineas, P., Mahoney, M., Muthukrishnan, S. & Sarlos, T. (2011). Faster least squares approximation. *Numerische Mathematik*, **117**: 219-249.

[7] FAN, J., HAN, F. AND LIU, H. (2013). Challenges of big data analysis. *arXiv:1308.1479.*

[8] KLEINER, A., TALWALKAR, A., SARKAR, P. AND JORDAN, M. I. (2014). A scalable bootstrap for massive data. *J. R. Stat. Soc. Series B Stat. Methodol.* **76**(4), 795-816.

[9] LIANG, F., CHENG, Y., SONG, Q., PARK, J., AND YANG, P. (2013). Stochastic approximation method for analysis of large geostatistical data. *J. Amer. Statist. Assoc.* **108**(501): 325-339.

[10] MA, P. AND SUN, X. (2014). Leveraging for big data regression. *Computational Statistics*, **7** (1): 70-76.

[11] MA, P. , MAHONEY, M.W, AND YU, B. (2015). A statistical perspective on algorithmic leveraging. *Journal of Machine Learning Research*, **16** (April): 861-911.

[12] MAHONEY, M. W. (2011). Randomized algorithms for matrices and data. *arXiv:1104.5557v3* [cs.DS]

[13] MCCULLAGH, P. AND NELDER, J. (1984). *Generalized Linear Models*. Springer-Verlag, New York, NY.

[14] O'Sullivan, F. (1988). Fast computation of fully automated log-density and log-hazard estimators. *SIAM Journal of Scientic and Statistical Computation*, **9**: 363-379.

[15] WANG, H., YANG, M. and STUFKEN, J. (2019). Information-Based Optimal Subdata Selection for Big Data Linear Regression. *J. Amer. Statist. Assoc.* **114** (525): 393-405.

[16] Wang, H., Zhu, R. and Ma, P. (2018). Optimal subsampling for large sample logistic regression. *J. Amer. Statist. Assoc.* **113**(522): 829-844.

[17] XU, P., YANG, J., ROOSTA-KHORASANI, F., RÉ, C. AND MAHONEY, M.W. (2016). Subsampled Newton Methods with Non-uniform Sampling. *arXiv:1607.00559.v2* [math.OC].

[18] Zhang, S., Tan, F. and Peng, H. (2023). Sample Size Determination for Multidimensional Parameters and A-Optimal Subsampling in a Big Data Linear Regression Model. *Preprint*, Available at `http://math.iupui.edu/~feitan/ZTP_SSD-23(002).pdf`