Schroeder's Equation in Several Variables

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Includes recent work of Ruth Enoch, Arkansas Tech University,

and of Robert A. (Bobby) Bridges and Maria Neophytou,

Purdue University

If φ is an analytic map of $\mathbb{D}$ into itself, *Schroeder's functional equation* is

$$f \circ \varphi = \lambda f$$

for f analytic on $\mathbb{D}$, not identically 0, and λ a complex number.

Theorem (Koenigs, 1884). *Let φ be an analytic map of the disk into itself, not an automorphism of the disk, with $\varphi(0) = 0$. If $\varphi'(0) = 0$, then only solutions of Schroeder's equation are $\lambda = 1$ and f constant.*

If $\varphi'(0) \neq 0$, then there is a unique function σ analytic on the disk with $\sigma'(0) = 1$ and

$$\sigma \circ \varphi = \varphi'(0)\sigma$$

Moreover, in this case, the only solutions of Schroeder's functional equation are $\lambda = \varphi'(0)^k$ and $f = c\sigma^k$.

Outline of Proof. Let f be analytic on the disk and λ is a number so that

$$f \circ \varphi = \lambda f$$

Since φ maps $\mathbb{D}$ into itself, $\varphi(0) = 0$, for $a_1 = \varphi'(0)$, we get $|a_1| < 1$ and

$$\varphi(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

and

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$$

Putting these series into Schroeder's equation, we get

$$c_0 + c_1(a_1 z + a_2 z^2 + \cdots) + c_2(a_1 z + a_2 z^2 + \cdots)^2 + \cdots$$
$$= \lambda c_0 + \lambda c_1 z + \lambda c_2 z^2 + \cdots$$

Equating coefficients shows the only possible solutions are the ones in the conclusion of Koenig's Theorem, and, since this infinite collection of equations can be solved inductively, we get a *formal power series* for f.

BUT, do these formal power series converge in the disk?

Koenigs used the power series to show that the only solutions possible are (a one dimensional space of solutions) when $\lambda = \varphi'(0)^k$ where k is a non-negative integer.

To get the existence, Koenigs showed that

$$\sigma(z) = \lim_{n\to\infty} \frac{\varphi(\varphi(\cdots(\varphi(z)\cdots)))}{a_1^n}$$

converges uniformly on compact sets in the disk to an analytic function with $\sigma(0) = 0$, $\sigma'(0) = 1$, and

$$\sigma \circ \varphi = a_1 \sigma$$

that is, σ is a solution of Schroeder's equation for $\lambda = \varphi'(0)$. ∎

Various extensions of this result were made in century after Koenigs' work.

These extensions can be combined to give a single perspective on the solution of this problem:

Model for iteration of functions mapping the unit disk into itself.

Let φ be an analytic map of the unit disk $\mathbb{D}$ into itself,
not an automorphism of the disk.

Suppose that either φ does not have a fixed point in $\mathbb{D}$
or that $\varphi'(a) \neq 0$ for the fixed point a in $\mathbb{D}$.

Then there is a domain Δ, either the plane or a half-plane,
an automorphism Φ of Δ onto Δ,
and a mapping σ of $\mathbb{D}$ into Δ such that

$$\sigma \circ \varphi = \Phi \circ \sigma$$

Four distinct cases in the model:

If φ has a fixed point in $\mathbb{D}$:

- *(plane/dilation)* $\Delta = \mathbb{C}$, $\Phi(z) = \lambda z$ (Koenigs, Schroeder Eqn.)

If φ has no fixed points in $\mathbb{D}$:

- *(half-plane/dilation)* $\Delta = \{\text{Re } z > 0\}$, $\Phi(z) = \lambda z$ (Schroeder Eqn.)
- *(plane/translation)* $\Delta = \mathbb{C}$, $\Phi(z) = z + 1$ (Abel Eqn.)
- *(half-plane/translation)*

  $\Delta = \{\text{Im } z > 0\}$, $\Phi(z) = z \pm 1$ (Abel Eqn.)

A fifth case: φ has fixed point a in $\mathbb{D}$ with $\varphi'(a) = 0$ *cannot be* a solution of this kind of equation with Φ linear fractional, that is, the **Model** does not apply in this case.

Some applications of the model:

- Determination of the eigenvectors and eigenvalues of composition operators on spaces (e.g. $H^2(\mathbb{D})$, $A^2(\mathbb{D})$) of analytic functions on the disk
- Better understanding of iteration of the function φ, including questions about embeddability of the discrete semi-group of iterates of φ into a continuous semi-group
- Determination of the functions ψ mapping the disk into the disk that satisfy

$$\psi \circ \varphi = \varphi \circ \psi$$

Long Term Goal:

Find a similar model for iteration of analytic maps φ of unit ball in $\mathbb{C}^N$ into itself to do these things in the several variable setting.

If φ is an analytic map of B_N into itself, we must first decide 'What is Schroeder's functional equation in several variables?'

For the disk, in Schroeder's equation $f \circ \varphi = \lambda f$, Was f a map of the domain into the complex plane? or Was f a map of the domain into the $\mathbb{C}^N$ that contains the domain?

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First case: a literal eigenvalue equation, λ a number.

Second case: a linearizing change of variables, λ should be $N \times N$ matrix.

Two cases interdependent, but first is easier consequence of the second, so

Schroeder's functional equation is the equation

$$f \circ \varphi = Af$$

for f an analytic map of B_N into $\mathbb{C}^N$ and A an $N \times N$ matrix.

General Schroeder Problem.

For φ an analytic map of B_N into itself with $\varphi(0) = 0$, find necessary and sufficient conditions on $N \times N$ matrices A so that

$$f \circ \varphi = Af$$

has non-zero solutions f that are analytic maps of B_N into $\mathbb{C}^N$, and for each allowable A, find all such solutions.

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A More Focused Schroeder Problem.

For φ an analytic map of B_N into itself such that $|\varphi(z)| < |z|$ when $0 < |z| < 1$ and has $\varphi'(0)$ invertible,

find all functions that satisfy

$$f \circ \varphi = \varphi'(0) f$$

where f is an analytic map of B_N into $\mathbb{C}^N$ with $f'(0)$ invertible.

Example 1. Suppose φ is a linear fractional map of B_N into itself of the form

$$\varphi(z) = (\langle z, C\rangle + 1)^{-1} Az$$

where A has no eigenvalues of modulus 1. If P satisfies $(I - A^*)P = C$, then

$$f(z) = (\langle z, P\rangle + 1)^{-1} z$$

satisfies $f \circ \varphi = \varphi'(0) f$ and $f'(0) = I$.

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Example 2. If c_1 and c_2 are small and non-zero, then

$$\varphi(z_1, z_2) = (c_1 z_1, c_1^3 z_2 + c_2 z_1^2)$$

is an analytic map of B_2 into itself with $\varphi'(0)$ invertible. Then

$$f(z_1, z_2) = (z_1, z_2 + \frac{c_2 z_1^2}{c_1^3 - c_1^2}) \quad \text{and} \quad g(z_1, z_2) = (z_1, z_1^3)$$

are both solutions of Schroeder's functional equation, but $f'(0) = I$ and $g'(0)$ is rank 1.

Example 3.

$$\varphi(z_1, z_2) = (\frac{1}{2}z_1, \frac{1}{4}z_2 + \frac{1}{2}z_1^2)$$

It is not difficult to show that φ maps B_2 into itself and

$$\varphi'(0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}$$

which is clearly invertible.

However, examining the Taylor series for functions that are potential solutions of Schroeder's functional equation, we find there are none with $f'(0)$ invertible!

Definition.

Let φ be an analytic map of B_N into itself such that $|\varphi(z)| < |z|$ when $0 < |z| < 1$ and that $\varphi'(0)$ is invertible.

We say that φ *has resonance* if the eigenvalues of $\varphi'(0)$ are $\lambda_1, \lambda_2, \cdots, \lambda_k$ and there is j with $1 \leq j \leq k$ such that

$$\lambda_j = \lambda_1^{p_1} \lambda_2^{p_2} \cdots \lambda_k^{p_k}$$

where $p_1, p_2, \cdots, p_k$ are non-negative integers with $\sum_{\ell=1}^{k} p_\ell \geq 2$ and $p_j = 0$. In this case, we say λ_j *is a resonant eigenvalue of* $\varphi'(0)$.

Notice that in Example 3, in which $\varphi'(0)$ had eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = 1/4$, the function φ has resonance and λ_2 is a resonant eigenvalue because

$$\lambda_2 = \lambda_1^2 \lambda_2^0$$

Some History.

Many people have tried to generalize Koenigs' solution to several variables, and all have failed. Most of these attempts did not exclude the cases where there IS no solution!

Some people have known of the arithmetic obstruction to the solution for a long time, but it has not been know to all who have thought about the problem.

Local solutions under various hypotheses have been known for some time.

MacCluer and C. (2003) proved the existence of analytic Schroeder's equation solutions for functions φ with a specific large, finite matrix being diagonalizable. This condition always holds if $\varphi'(0)$ is diagonalizable and φ does not have resonance, and sometimes holds if φ does have resonance.

Some History (cont'd).

Enoch (2004) showed formal power series solutions exist in cases without resonance and gave necessary and sufficient conditions for formal power series solutions in cases with resonance to have solutions in terms of some conditions on the entries of a certain matrix.

Bridges (2011) solves the problem!

Proposition.

If T is lower triangular matrix that gives compact operator on $\mathcal{H}$ and $\rho > 0$, there is a positive integer m so that

$$T = \begin{pmatrix} U & 0 \\ V & W \end{pmatrix}$$

where U is an $m \times m$ matrix and $\|W\| \leq \rho$.

Corollary.

Suppose T is as in the Proposition and u is in $\mathbb{C}^m$ such that $Uu = \lambda u$ where $|\lambda| > \rho$. Then there is v in $\mathcal{H}$ so that $Tv = \lambda v$.

Proof: We want x so that

$$\begin{pmatrix} U & 0 \\ V & W \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} = \lambda \begin{pmatrix} u \\ x \end{pmatrix}$$

that is, we want $Vu + Wx = \lambda x$. Take $x = -(W - \lambda I)^{-1} V u$. ■

Definition.

Let φ be an analytic map of B_N into itself. If $\mathcal{H}$ is a Hilbert space of analytic functions on B_N, the *composition operator* C_φ *on* $\mathcal{H}$ is the operator given by

$$C_\varphi f = f \circ \varphi$$

Theorem. (MacCluer & C., 2003)

If φ is an analytic map of B_N into itself such that $|\varphi(z)| < |z|$ when $0 < |z| < 1$ and $G(r) = \exp(-1/(1-r))$, then C_φ is compact on the weighted Bergman space $A^2_G(B_N)$ which is given by

$$A^2_G(B_N) = \{f : B_N \to \mathbb{C} : f \text{ analytic and } \int_{B_N} |f|^2 G(|z|)\, d\nu_N(z) < \infty\}$$

where $d\nu_N$ is normalized volume measure on B_N.

Without loss of generality, we may assume $\varphi'(0)$ is upper triangular and this means the matrix for C_φ is lower triangular.

Theorem.

Suppose $\mathcal{H}$ is a Hilbert space of analytic functions defined on B_N taking values in $\mathbb{C}$ for which the monomials in the coordinate functions are an orthogonal basis.

If φ is an analytic map of B_N into itself such that $\varphi(0) = 0$ and C_φ is compact on $\mathcal{H}$, then the spectrum of C_φ is 0 together with

$$\{\lambda : \lambda = \lambda_1^{p_1} \lambda_2^{p_2} \cdots \lambda_k^{p_k} \text{ for } p_j \text{ non-negative integers and } \lambda_j \text{ eigenvalues of } \varphi'(0)\}$$

Proof: Eigenvectors for C_φ^* can be computed explicitly and theory of compact operators gives result. ∎

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$$\{\lambda : \lambda = \lambda_1^{p_1}\lambda_2^{p_2}\cdots\lambda_k^{p_k} \text{ for } p_j \text{ non-negative integers and } \lambda_j \text{ eigenvalues of } \varphi'(0)\}$$

Corollary.

For φ as above, for each λ that is a finite product of eigenvalues of $\varphi'(0)$, there is a function, f, analytic on B_N such that $f \circ \varphi = \lambda f$.

Proof: Theorem gives an eigenvector in $\mathcal{H}$, but all vectors in $\mathcal{H}$ are analytic functions on B_N! ∎

Main Theorem. (R. A. Bridges, 2011)

Let φ be an analytic map of B_N into itself such that $|\varphi(z)| < |z|$ when $0 < |z| < 1$ and suppose $\varphi'(0)$ is invertible and upper triangular.

Writing C_φ for the matrix of composition with φ in the basis of monomials, choose a block representation such that

$$C_\varphi = \begin{pmatrix} U & 0 \\ V & W \end{pmatrix}$$

so that no resonant eigenvalues of $\varphi'(0)$ are on the diagonal of W.

Writing C_φ for the matrix of composition with φ as

$$C_\varphi = \begin{pmatrix} U & 0 \\ V & W \end{pmatrix}$$

the matrix U is an $m \times m$ matrix with $m \geq N+1$ and has block form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varphi'(0)^t & 0 \\ 0 & U_1 & U_2 \end{pmatrix}$$

There is an analytic map F of B_N into $\mathbb{C}^N$ satisfying

$$F \circ \varphi = \varphi'(0) F$$

and having $F'(0)$ invertible if and only if

every eigenvector x of $\varphi'(0)^t$ corresponds to an eigenvector $\begin{pmatrix} 0 \\ x \\ y \end{pmatrix}$ for U.

Other results of Bridges:

Theorem.

If φ is an analytic map of B_N into itself such that $|\varphi(z)| < |z|$ when $0 < |z| < 1$, then there is an analytic map F of B_N into $\mathbb{C}^N$ satisfying

$$F \circ \varphi = \varphi'(0)F$$

such that the component functions of F are linearly independent.

Theorem.

If φ is an analytic map of B_N into itself such that $|\varphi(z)| < |z|$ when $0 < |z| < 1$ and $k > 1$ is an integer, then there is an analytic map G of B_N into $\mathbb{C}^N$ satisfying

$$G \circ \varphi = \varphi'(0)^k G$$

such that the component functions of G are linearly independent, but no such solution G has $G'(0)$ invertible.

References.

Robert A. Bridges, A Solution to Schroeder's Equation in Several Variables, (2011), arXiv:1106.3370

C. C. Cowen & B. D. MacCluer, Schroeder's Equation in Several Variables, Taiwanese J. Math. 7(2003)129-154.

http://www.math.iupui.edu/˜ccowen/Downloads.html