Composition Operators
Weighted Composition Operators,
and Applications to Operator Theory

Carl C. Cowen

IUPUI

(Indiana University Purdue University Indianapolis)

North British Functional Analysis Seminar

Lancaster University, 3 June 2013

Composition Operators
Weighted Composition Operators,
and Applications to Operator Theory

Carl C. Cowen

IUPUI

(Indiana University Purdue University Indianapolis)

North British Functional Analysis Seminar

Lancaster University, 3 June 2013

Joint work with Gajath Gunatillake, Univ. Sharjah

and with Eungil Ko, Ewha Women's University

Some History & Philosophy

From the beginning, functional analysis was based on vector spaces of “functions” and such spaces are still among the most important

Concrete functional analysis developed with results on spaces

of integrable functions, with special classes of differential operators,

and sometimes used better behaved inverses of differential operators

The abstraction of these ideas led to:

Banach and Hilbert spaces

Bounded operators, unbounded closed operators, compact operators

For me, functional analysis is mostly study of operators on separable Hilbert spaces and the study of illuminating examples is a critical tool in creating more abstract theories.

In particular, I want my work to foster the interaction between

Concrete examples and the development of theory

For example:

Spectral theory as a generalization of Jordan form and diagonalizability

Hermitian, normal, multiplication, subnormal operators
as extensions of diagonalization of matrices

Unilateral Shift operators as an examples of asymmetric behavior possible
in operators on infinite dimensional spaces

Studying *composition operators* can be seen as extension of this process

The classical Banach spaces are spaces of functions on a set X:

if φ maps X *into* itself, and ψ maps space X into complex numbers, $\mathbb{C}$

we can imagine a *composition operator* with symbol φ,

$$C_\varphi f = f \circ \varphi$$

and a *weighted composition operator* with symbols ψ and φ,

$$W_{\psi,\varphi} f = \psi \cdot f \circ \varphi$$

for f in the Banach space.

These operators are formally linear:

$$\psi \cdot (af + bg) \circ \varphi = a\psi \cdot f \circ \varphi + b\psi \cdot g \circ \varphi$$

But other properties, like "Are $f \circ \varphi$ or $\psi \cdot f \circ \varphi$ in the space X?"

clearly depend on the maps φ and ψ and the Banach space of functions.

Some Examples

Several classical operators are composition operators. For example, we may regard $\ell^p(\mathbb{N})$ as the space of functions of $\mathbb{N}$ into $\mathbb{C}$ that are p^{th} power integrable with respect to counting measure by thinking x in ℓ^p as the function $x(k) = x_k$. If $\varphi : \mathbb{N} \to \mathbb{N}$ is given by $\varphi(k) = k + 1$, then $(C_\varphi x)(k) = x(\varphi(k)) = x(k+1) = x_{k+1}$, that is,

$$C_\varphi : (x_1, x_2, x_3, x_4, \cdots) \mapsto (x_2, x_3, x_4, x_5, \cdots)$$

so C_φ is the "backward shift".

In fact, backward shifts of all multiplicities can be represented as composition operators.

Moreover, composition operators often come up in studying other operators. For example, if we think of the operator of multiplication by z^2,

$$(M_{z^2}f)(z) = z^2 f(z)$$

Easy to see M_{z^2} commutes with multiplication operators and C_{-z}

$$(M_{z^2}C_{-z}f)(z) = M_{z^2}f(-z) = z^2 f(-z)$$

and

$$(C_{-z}M_{z^2}f)(z) = C_{-z}(z^2 f(z)) = (-z)^2 f(-z) = z^2 f(-z)$$

In some contexts, set of operators commuting with M_{z^2} is algebra generated by multiplication operators and composition operator C_{-z}.

Forelli: Isometries of H^p, $p \neq 2$, $p > 1$, are weighted composition ops

Lomonosov (1973):

If an operator A commutes with an operator $B \neq \lambda I$ that commutes with a compact $C \neq 0$, then A has a non-trivial invariant subspace.

Question: *Did Lomonosov solve the invariant subspace problem?*

My interest in composition operators comes from my thesis:

Goal: Show that Lomonosov did not solve the ISP; specifically:

show that M_z on H^2 does not satisfy Lomonosov's hypothesis.

Our Context

Some Hilbert spaces of analytic functions on unit disk, $\mathbb{D} = \{z : |z| < 1\}$:

(1) Hardy Hilbert space:

$$H^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : f(z) = \sum_{n=0}^{\infty} a_n z^n \ \text{ with } \ \|f\|_{H^2}^2 = \sum |a_n|^2 < \infty\}$$

where for f and g in $H^2(\mathbb{D})$, we have $\langle f, g\rangle = \sum a_n \overline{b_n}$

(2) Bergman Hilbert space:

$$A^2(\mathbb{D}) = A^2 = \{f \text{ analytic in } \mathbb{D} : \ \|f\|_{A^2}^2 = \int_{\mathbb{D}} |f(\zeta)|^2 \frac{dA(\zeta)}{\pi} < \infty\}$$

where for f and g in $A^2(\mathbb{D})$, we have $\langle f, g\rangle = \int f(\zeta)\overline{g(\zeta)}\, dA(\zeta)/\pi$

(3) Standard weight Bergman Hilbert spaces, for $\kappa > 1$:

$$A^2_{\kappa-2} = \{f \text{ analytic in } \mathbb{D} : \ \|f\|^2 = \int_{\mathbb{D}} |f(\zeta)|^2 (\kappa-1)(1-|\zeta|^2)^{\kappa-1} \frac{dA(\zeta)}{\pi} < \infty\}$$

In this context, the study of composition operators was initiated 40+ years ago by Nordgren, Schwartz, Rosenthal, Caughran, Kamowitz, and others.

My perspective:

Studying composition, weighted composition operators can solve interesting concrete problems and illuminate new parts of Operator Theory generally.

Goal: Understand composition operators

Relate properties of functions φ, ψ to properties of operators C_φ and $W_{\psi,\varphi}$

Goal: Use composition ops to understand other operators

Relate properties of operators C_φ and $W_{\psi,\varphi}$ to properties of other operators

Example: **Conjecture:**

If ψ is a bounded analytic on $\mathbb{D}$, set of operators commuting with M_ψ on Hardy and Bergman spaces above is algebra generated by analytic multiplication operators and composition operators that commute with M_ψ

Kernel Functions

On a functional Hilbert space, $\mathcal{H}$, on the unit disk,

the linear functional $f \mapsto f(\alpha)$ for f in $\mathcal{H}$ and α in $\mathbb{D}$ is bounded.

For a point α in the disk $\mathbb{D}$, the kernel function K_α is the function in $\mathcal{H}$ such that for all f in $\mathcal{H}$, we have

$$\langle f, K_\alpha \rangle = f(\alpha)$$

The Bergman spaces $A^2_{\kappa-2}$ for $\kappa > 1$ and

the Hardy space H^2 (where $\kappa = 1$) are all functional Hilbert spaces with

$$K_\alpha(z) = \frac{1}{(1 - \overline{\alpha}z)^\kappa}$$

Indeed, this can be used as the definitions of these spaces.

For the remainder of this presentation, for $\kappa \geq 1$,

we will write $\mathcal{H}_\kappa$ for one of these Hardy or Bergman spaces, that is:

$\mathcal{H}_\kappa$ is the functional Hilbert space with kernel function $K_\alpha = (1 - \overline{\alpha}z)^{-\kappa}$

Some Properties of Composition Operators

When an operator theorist studies an operator for the first time, questions are asked about the boundedness and compactness of the operator,

about norms,

spectra,

and adjoints.

While the whole story is not known, much progress has been made $\cdots$

and we expect the answers to be given in terms of analytic and geometric properties of φ.

Very often, calculations with kernel functions give ways to connect analytic and geometric properties of φ, ψ with operator properties of C_φ and $W_{\psi,\varphi}$.

For a point α in the disk $\mathbb{D}$,

because the kernel function K_α is a function in $\mathcal{H}_\kappa$, we have

$$\|K_\alpha\|^2 = \langle K_\alpha, K_\alpha\rangle = K_\alpha(\alpha) = \frac{1}{(1-\overline{\alpha}\alpha)^\kappa} = \frac{1}{(1-|\alpha|^2)^\kappa}$$

For each f in $\mathcal{H}_\kappa$ and α in the disk,

$$\langle f, W_{\psi,\varphi}^* K_\alpha\rangle = \langle W_{\psi,\varphi} f, K_\alpha\rangle = \langle \psi f\circ\varphi, K_\alpha\rangle = \psi(\alpha) f(\varphi(\alpha)) = \langle f, \overline{\psi(\alpha)} K_{\varphi(\alpha)}\rangle$$

Since this is true for every f, we see $W_{\psi,\varphi}^*(K_\alpha) = \overline{\psi(\alpha)} K_{\varphi(\alpha)}$

and $C_\varphi^*(K_\alpha) = K_{\varphi(\alpha)}$

Further exploitation of this line of thought shows that C_φ is invertible if and only if φ is an automorphism of the disk and in this case, $C_\varphi^{-1} = C_{\varphi^{-1}}$

Theorems from Complex Analysis

Theorem: (Littlewood Subordination Theorem)

Let φ be an analytic map of the unit disk into itself such that $\varphi(0) = 0$. If G is a subharmonic function in $\mathbb{D}$, then for $0 < r < 1$

$$\int_0^{2\pi} G(\varphi(re^{it\theta})\, d\theta \leq \int_0^{2\pi} G(re^{it\theta})\, d\theta$$

For H^2, the Littlewood subordination theorem plus some easy calculations for changes of variables induced by automorphisms of the disk yields the following estimate of the norm for composition operators on H^2:

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\frac{1}{2}} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{1}{2}}$$

and similar estimates for the norms of C_φ on $\mathcal{H}_\kappa$.

On all $\mathcal{H}_\kappa$, the operators C_φ are bounded for all functions φ that are analytic and map $\mathbb{D}$ into itself

Not true for all Hilbert spaces of analytic functions on the disk:

If the function z is in $\mathcal{H}$, and C_φ is bounded on $\mathcal{H}$, then $C_\varphi z = \varphi$ is in $\mathcal{H}$.

For some maps φ of the disk into itself, φ is *not* a vector in the Dirichlet space, so C_φ is not bounded on the Dirichlet space for such φ.

This is the sort of result we seek, connecting the properties of the operator C_φ with the analytic and geometric properties of φ.

We will see that many results about composition operators show that the behavior of C_φ depends on the fixed points of φ. Digress to talk about fixed points.

If φ is a continuous map of $\overline{\mathbb{D}}$ into $\overline{\mathbb{D}}$, then φ must have a fixed point in $\overline{\mathbb{D}}$.

But, we *only* assume φ is analytic on $\mathbb{D}$, open disk!

Definition

Suppose φ is an analytic map of $\mathbb{D}$ into itself.

If $|b| < 1$, we say b *is a fixed point of* φ if $\varphi(b) = b$.

If $|b| = 1$, we say b *is a fixed point of* φ if $\lim_{r\to 1^-} \varphi(rb) = b$.

Julia-Caratheordory Theorem implies

If b is a fixed point of φ with $|b| = 1$, then $\lim_{r\to 1^-} \varphi'(rb)$ exists (call it $\varphi'(b)$) and $0 < \varphi'(b) \leq \infty$.

Denjoy-Wolff Theorem (1926)

If φ is an analytic map of $\mathbb{D}$ into itself, not the identity map, there is a unique fixed point, a, of φ in $\overline{\mathbb{D}}$ such that $|\varphi'(a)| \leq 1$.

For φ not an elliptic automorphism of $\mathbb{D}$, for each z in $\mathbb{D}$, the sequence

$$\varphi(z),\ \varphi_2(z) = \varphi(\varphi(z)),\ \varphi_3(z) = \varphi(\varphi_2(z)),\ \varphi_4(z) = \varphi(\varphi_3(z)),\ \cdots$$

converges to a and the convergence is uniform on compact subsets of $\mathbb{D}$.

This distinguished fixed point will be called the *Denjoy-Wolff point of* φ.

The Schwarz-Pick Lemma implies φ has at most one fixed point in $\mathbb{D}$ and if φ has a fixed point in $\mathbb{D}$, it must be the Denjoy-Wolff point.

Examples

(1) $\varphi(z) = (z + 1/2)/(1 + z/2)$ is an automorphism of $\mathbb{D}$ fixing 1 and -1.

The Denjoy-Wolff point is $a = 1$ because $\varphi'(1) = 1/3$ (and $\varphi'(-1) = 3$)

(2) $\varphi(z) = z/(2 - z^2)$ maps $\mathbb{D}$ into itself and fixes 0, 1, and -1.

The Denjoy-Wolff point is $a = 0$ because $\varphi'(0) = 1/2$ (and $\varphi'(\pm 1) = 3$)

(3) $\varphi(z) = (2z^3 + 1)/(2 + z^3)$ is an inner function fixing fixing 1 and -1 with Denjoy-Wolff point $a = 1$ because $\varphi'(1) = 1$ (and $\varphi'(-1) = 9$)

(4) Inner function $\varphi(z) = \exp(z + 1)/(z - 1)$ has a fixed point in $\mathbb{D}$, Denjoy-Wolff point $a \approx .21365$, and infinitely many fixed points on $\partial\mathbb{D}$

Denjoy-Wolff Thm suggests looking for a model for iteration of maps of $\mathbb{D}$

Five different types of maps of $\mathbb{D}$ into itself from the perspective of iteration,

classified by the behavior of the map near the Denjoy-Wolff point, a

In one of these types, $\varphi'(a) = 0$, (e.g., $\varphi(z) = (z^2 + z^3)/2$ with $a = 0$),

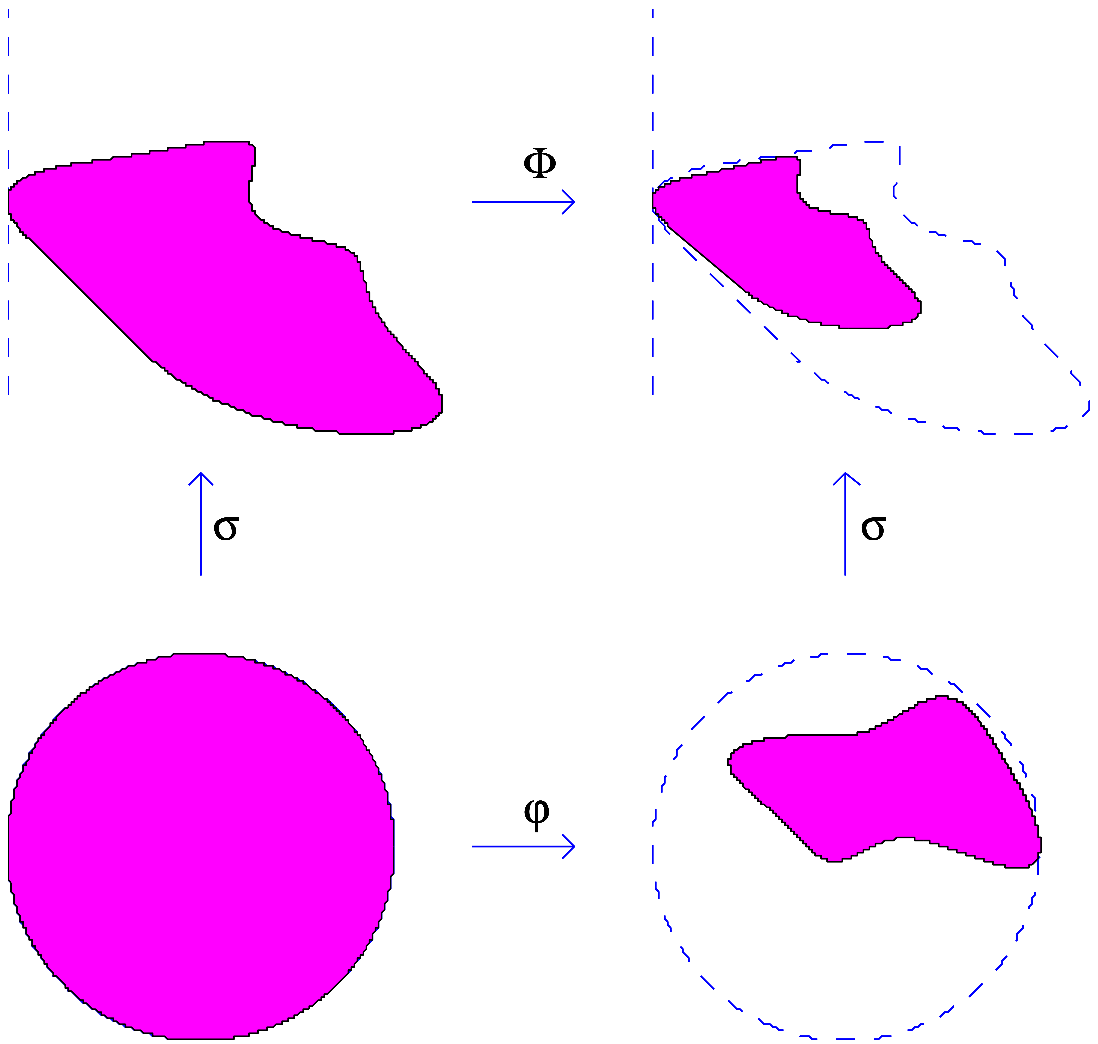
the model for iteration not yet useful for studying composition operators

In the other four types, when $\varphi'(a) \neq 0$, the map φ can be intertwined with

a linear fractional map and classified by the possible type of intertwining:

σ intertwines Φ and φ in the equality $\Phi \circ \sigma = \sigma \circ \varphi$

We want to do this with Φ linear fractional and σ univalent near a, so that σ is, locally, a change of variables. Using the notion of fundamental set, this linear fractional model becomes essentially unique [Cowen, 1981]



A linear fractional model in which φ maps $\mathbb{D}$ into itself with $a = 1$ and

$\varphi'(1) = \dfrac{1}{2}$, σ maps $\mathbb{D}$ into the right half plane, and $\Phi(w) = \dfrac{1}{2}\,w$

Linear Fractional Models:

- φ maps $\mathbb{D}$ into itself with $\varphi'(a) \neq 0$ (φ not an elliptic automorphism)
- Φ is a linear fractional automorphism of Ω onto itself
- σ is a map of $\mathbb{D}$ into Ω with $\quad \Phi \circ \sigma = \sigma \circ \varphi$

I. (plane dilation) $|a| < 1$, $\Omega = \mathbb{C}$, $\sigma(a) = 0$, $\Phi(w) = \varphi'(a)w$

II. (half-plane dilation) $|a| = 1$ with $\varphi'(a) < 1$, $\Omega = \{w : \operatorname{Re} w > 0\}$,

$\sigma(a) = 0$, $\Phi(w) = \varphi'(a)w$

III. (plane translation) $|a| = 1$ with $\varphi'(a) = 1$, $\Omega = \mathbb{C}$, $\Phi(w) = w + 1$

$\{\varphi_n(0)\}$ NOT an interpolating sequence

IV. (half-plane translation) $|a| = 1$ with $\varphi'(a) = 1$, $\Omega = \{w : \operatorname{Im} w > 0\}$,

(or $\Omega = \{w : \operatorname{Im} w < 0\}$), $\Phi(w) = w + 1$

$\{\varphi_n(0)\}$ IS an interpolating sequence

Compactness

In addition to asking "When is C_φ bounded?" operator theorists would want to know "When is C_φ compact?"

Because

- analytic functions take their maxima at the boundary
- compact operators should take most vectors to much smaller vectors

expect C_φ compact implies $\varphi(\mathbb{D})$ is far from the boundary in some sense.

If $m(\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}) > 0$, then C_φ is not compact.

If $\|\varphi\|_\infty < 1$, then C_φ is compact.

In $\mathcal{H}_\kappa$ and similar spaces, as $|\alpha| \to 1$, then $\frac{1}{\|K_\alpha\|}K_\alpha \to 0$ weakly.

C_φ is compact if and only if C_φ^* is compact, and in this case, we must have

$$\left\| C_\varphi^* \left(\frac{1}{\|K_\alpha\|} K_\alpha \right) \right\| = \frac{\|K_{\varphi(\alpha)}\|}{\|K_\alpha\|} = \left(\frac{1 - |\alpha|^2}{1 - |\varphi(\alpha)|^2} \right)^{\kappa/2}$$

is going to zero.

Now if $\alpha \to \zeta$ non-tangentially with $|\zeta| = 1$ and the angular derivative $\varphi'(\zeta)$ exists, then the Julia-Caratheodory Theorem shows that $\frac{1-|\alpha|^2}{1-|\varphi(\alpha)|^2} \to \frac{1}{\varphi'(\zeta)}$

In particular, C_φ compact implies no angular derivative of φ is finite.

Theorem:

For $\kappa > 1$ and φ an analytic map of the disk into itself, C_φ is compact on $\mathcal{H}_\kappa$ if and only if φ has no finite angular derivative.

Theorem (1987, J.H. Shapiro)

Suppose φ is an analytic map of $\mathbb{D}$ into itself. For C_φ acting on $H^2(\mathbb{D})$,

$$\|C_\varphi\|_e^2 = \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{-\log|w|}$$

where N_φ is the Nevanlinna counting function.

Corollary

C_φ is compact on $H^2(\mathbb{D})$ if and only if $\displaystyle\limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{-\log|w|} = 0$

Caughran and Schwartz (1975) showed that if C_φ is compact on H^2, and found spectrum of C_φ.

Generalized Caughran-Schwartz Theorem.

Let φ be analytic map on $\mathbb{D}$ with D.W. point a, let $\kappa \geq 1$, and suppose C_φ is compact on $\mathcal{H}_\kappa$.

Then $|a| < 1$ and the spectrum of C_φ is

$$\sigma(C_\varphi) = \{0, 1\} \cup \{\varphi'(a)^n : n = 1, 2, 3, \cdots\}$$

Proof:

Without loss of generality, $\varphi(0) = 0$.

The monomials $1, z, z^2, \cdots$ form an orthogonal basis for $\mathcal{H}_\kappa$ and we will consider the matrix for C_φ with respect to this basis.

Proof (cont'd):

Without loss of generality, $\varphi(0) = 0$.

Since $C_\varphi 1 = 1 \circ \varphi = 1$, the column of the matrix for C_φ corresponding to the basis vector 1 is $(1, 0, 0, \cdots)$. Similarly, column of the the matrix for C_φ corresponding to the basis vector z^k is the vector of Taylor coefficients of $C_\varphi z^k = \varphi^k$ which is $(0, 0, \cdots 0, \varphi'(0)^k, k\varphi'(0)^{k-1}\varphi''(0)/2, \cdots)$

In particular, the matrix for C_φ is lower triangular which means the matrix for C_φ^* is upper triangular

Proof (cont'd):

Without loss of generality, $\varphi(0) = 0$.

Since $C_\varphi 1 = 1 \circ \varphi = 1$, the column of the matrix for C_φ corresponding to the basis vector 1 is $(1, 0, 0, \cdots)$. Similarly, column of the the matrix for C_φ corresponding to the basis vector z^k is the vector of Taylor coefficients of $C_\varphi z^k = \varphi^k$ which is $(0, 0, \cdots 0, \varphi'(0)^k, k\varphi'(0)^{k-1}\varphi''(0)/2, \cdots)$

In particular, the matrix for C_φ is lower triangular which means the matrix for C_φ^* is upper triangular

Triangularity of C_φ^* implies, for any positive integer n, as a block matrix

$$C_\varphi^* \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A is $n \times n$ upper triangular and the lower left is a 0 matrix

Proof (cont'd):

The compactness of C_φ^* implies, for sufficiently large n, as a block matrix

$$C_\varphi^* \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

and $\|D\|$ can be made as small as we like.

As a consequence, each of the non-zero eigenvalues of C_φ^* is an eigenvalue of an upper left corner, A, for sufficiently large n, and every eigenvalue of such an A is an eigenvalue of C_φ^*.

Note that

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \overline{\varphi'(0)} & * & * & \cdots & * \\ 0 & 0 & \overline{\varphi'(0)}^2 & * & \cdots & * \\ 0 & 0 & 0 & \overline{\varphi'(0)}^3 & \cdots & * \\ & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \overline{\varphi'(0)}^n \end{pmatrix}$$

We see that this means that the eigenvalues of A are 1, $\overline{\varphi'(0)}$, $\overline{\varphi'(0)}^2$,$\cdots$, and $\overline{\varphi'(0)}^n$, each of multiplicity one, and that therefore, the non-zero eigenvalues of C_φ^* are $\{\overline{\varphi'(0)}^k\}_{k=0}^\infty$, each with multiplicity one.

The spectral theory of compact operators therefore implies that the non-zero eigenvalues of C_φ are $\{\varphi'(0)^k\}_{k=0}^\infty$, each with multiplicity one, and $\sigma(C_\varphi) = \{0, 1\} \cup \{\varphi'(a)^n : n = 1, 2, 3, \cdots\}$. ∎

Since each of the numbers $\{\varphi'(0)^k\}_{k=0}^{\infty}$, is a non-zero eigenvalue with multiplicity one, for each of the numbers $\varphi'(0)^k$ there must be a one-dimensional subspace of $\mathcal{H}_\kappa$ (!!!) consisting of eigenvectors of C_φ.

A theorem of Koenigs tells us about the eigenvectors.

Theorem (Koenigs, 1884)

If φ is analytic map of $\mathbb{D}$ into itself, $\varphi(0) = 0$, and $0 < |\varphi'(0)| < 1$, then there is a unique map σ with $\sigma(0) = 0$, $\sigma'(0) = 1$, and $\sigma \circ \varphi = \varphi'(0)\sigma$

Moreover, if f is analytic (not the zero map) and λ is a number so that $f \circ \varphi = \lambda f$, then $\lambda = \varphi'(0)^n$ for some $n = 0, 1, 2, 3, \cdots$ and $f = c\,\sigma^n$ for some c.

This is Case I (plane-dilation) in the linear fractional model.

Theorem (Koenigs, 1884)

If φ is analytic map of $\mathbb{D}$ into itself, $\varphi(0) = 0$, and $0 < |\varphi'(0)| < 1$, then there is a unique map σ with $\sigma(0) = 0$, $\sigma'(0) = 1$, and $\sigma \circ \varphi = \varphi'(0)\sigma$

Moreover, if f is analytic (not the zero map) and λ is a number so that $f \circ \varphi = \lambda f$, then $\lambda = \varphi'(0)^n$ for some $n = 0, 1, 2, 3, \cdots$ and $f = c\,\sigma^n$ for some c.

Proof:

Starting with the second part, suppose f satisfies $f(\varphi(z)) = \lambda f(z)$ for some λ and all z in $\mathbb{D}$. Consider the Taylor series for f, $f(z) = \sum a_k z^k$ with first non-zero coefficient a_n, that is, $a_n \neq 0$ and $a_k = 0$ for k an integer, $k < n$. Since any non-zero multiple of f will work just as well, we suppose $a_n = 1$.

Compare the Taylor series for $f \circ \varphi$ and λf $\quad \cdots$

get f unique and $\lambda = \varphi'(0)^n$

Theorem (Koenigs, 1884)

If φ is analytic map of $\mathbb{D}$ into itself, $\varphi(0) = 0$, and $0 < |\varphi'(0)| < 1$, then there is a unique map σ with $\sigma(0) = 0$, $\sigma'(0) = 1$, and $\sigma \circ \varphi = \varphi'(0)\sigma$

Moreover, if f is analytic (not the zero map) and λ is a number so that $f \circ \varphi = \lambda f$, then $\lambda = \varphi'(0)^n$ for some $n = 0, 1, 2, 3, \cdots$ and $f = c\,\sigma^n$ for some c.

Proof:

For the first part, define σ by $\sigma(z) = \lim_{k\to\infty} \dfrac{\varphi_k(z)}{\varphi'(0)^k}$ where $\varphi_2(z) = \varphi(\varphi(z))$, $\varphi_3(z) = \varphi(\varphi_2(z))$, etc. Establish convergence; observe σ satisfies functional equation, $\sigma(0) = 0$, and $\sigma'(0) = 1$.

To finish, since the solution f from the earlier part was unique and σ^n also satisfies the conditions, $f = \sigma^n$.

Theorem (Koenigs, 1884)

If φ is analytic map of $\mathbb{D}$ into itself, $\varphi(0) = 0$, and $0 < |\varphi'(0)| < 1$, then there is a unique map σ with $\sigma(0) = 0$, $\sigma'(0) = 1$, and $\sigma \circ \varphi = \varphi'(0)\sigma$

New Proof (MacCluer, C.):

The hypotheses on φ guarantee a unique formal power series solution $\widetilde{\sigma}$ of the functional equation $\sigma \circ \varphi = \varphi'(0)\sigma$ for which $\widetilde{\sigma}'(0) = 1$.

Let $\mathcal{H}$ be a functional Hilbert space on $\mathbb{D}$ for which the $\{z^n\}$ are an orthogonal basis. Because $\varphi(0) = 0$, the matrix for C_φ^* with respect to this basis for $\mathcal{H}$ will be upper triangular with diagonal entries $\varphi'(0)^n$. This means that each of the numbers $\overline{\varphi'(0)}^n$ is an eigenvalue of multiplicity 1 for the operator C_φ^* acting on $\mathcal{H}$.

Theorem (Koenigs, 1884)

If φ is analytic map of $\mathbb{D}$ into itself, $\varphi(0) = 0$, and $0 < |\varphi'(0)| < 1$, then there is a unique map σ with $\sigma(0) = 0$, $\sigma'(0) = 1$, and $\sigma \circ \varphi = \varphi'(0)\sigma$

New Proof (MacCluer, C.) (cont'd):

For this function φ, there is a functional Hilbert space, $\mathcal{H}$, on the disk (a weighted Bergman space) for which $\{z^n\}$ is an orthorgonal basis that is large enough that C_φ is compact. This means that each of the numbers $\varphi'(0)^n$ is an eigenvalue of multiplicity 1 for the operator C_φ acting on $\mathcal{H}$.

Let f be the eigenvector of C_φ for the eigenvalue $\varphi'(0)$ that has $f'(0) = 1$. Because $\mathcal{H}$ is a vector space of analytic functions and f is in $\mathcal{H}$, the function f s an analytic function on the disk, and since $\widetilde{\sigma}$ is the unique formal power series solution of we must have $f = \widetilde{\sigma}$ is the analytic solution we seek.

Generalized Koenigs' Theorem (Bridges, 2012):

Suppose φ is an analytic map of B_N into itself, $\varphi(0) = 0$, and each eigenvalue, μ_j, of $\varphi'(0)$ satisfies $0 < |\mu_j| < 1$.

If σ is a formal power series in z_j, $j = 1, \cdots, n$ that is a solution of the functional equation $\sigma(\varphi(z)) = \varphi'(0)\sigma(z)$ with $\sigma(0) = 0$ and $\sigma'(0) = I$, then σ is actually analytic in B_N.

Proof:

For this function φ, there is a functional Hilbert space, $\mathcal{H}$, on the ball B_N (a weighted Bergman space) for which $\{z_j^n\}$ is an orthorgonal basis that is large enough that C_φ is compact. The above proof extends!

This is surprising because the Koenigs proof does *not* extend and there are sometimes arithmetic obstacles for the existence of such a σ!

Spectral Theory

Most of the results in spectral theory of composition operators come from the linear fractional models.

Theorem

Let φ be automorphism of $\mathbb{D}$ and a in $\overline{\mathbb{D}}$ the fixed point with $|\varphi'(a)| \leq 1$. Consider C_φ acting on H^2.

- *If $|a| < 1$ (φ is elliptic), then $\sigma(C_\varphi) = \overline{\{\varphi'(a)^n\}_{n=0}^{\infty}}$*
- *If $|a| = 1$ and $\varphi'(a) = 1$ (φ is parabolic), then $\sigma(C_\varphi) = \partial\mathbb{D}$*
- *If $|a| = 1$ and $\varphi'(a) < 1$ (φ is hyperbolic), then*

$$\sigma(C_\varphi) = \{\lambda : \sqrt{\varphi'(a)} \leq |\lambda| \leq \frac{1}{\sqrt{\varphi'(a)}}\}$$

Examples (also on H^2)

(1) (plane dilation) $\quad \varphi(z) = (1+i)z/2, \quad a = 0, \quad \varphi'(a) = (1+i)/2,$

C_φ compact

$$\sigma(C_\varphi) = \{0\} \cup \{ \left(\frac{1+i}{2}\right)^n : n = 0, 1, 2, \cdots\}$$

(2) (plane dilation) $\quad \varphi(z) = -z/2 + 1/2, \quad a = 1/3, \quad \varphi'(a) = -1/2,$

C_φ not compact ($\varphi(-1) = 1$), $\quad$ but $C_\varphi^2 = C_{\varphi\circ\varphi}$ is compact

$$\sigma(C_\varphi) = \{0\} \cup \{ \left(-\frac{1}{2}\right)^n : n = 0, 1, 2, \cdots\}$$

Examples (also on H^2) (cont'd)

(3) (plane dilation) $\varphi(z) = z/(2-z), \quad a = 0, \quad \varphi'(a) = 1/2,$

but also $\varphi(1) = 1$ and $\varphi'(1) = 2$, so C_φ not compact

$$\sigma(C_\varphi) = \{1\} \cup \{\lambda : |\lambda| \leq \frac{1}{\sqrt{2}}\}$$

(4) (half-plane dilation) $\varphi(z) = z/3 + 2/3, \quad a = 1, \quad \varphi'(a) = 1/3,$

so C_φ not compact

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \frac{1}{\sqrt{\varphi'(a)}}\} = \{\lambda : |\lambda| \leq \sqrt{3}\}$$

(5) (plane translation) $\varphi(z) = \dfrac{(2-t)z + t}{-tz + 2 + t}$ for $\mathrm{Re}\,t > 0, \quad a = 1, \quad \varphi'(1) = 1$

$$\sigma(C_\varphi) = \{e^{\beta t} : \beta \leq 0\} \cup \{0\}$$

The examples from the linear fractional maps give an indication of how the spectra vary depending on the case from the model for iteration – this dependence appears to persist throughout the study of composition operators on spaces of analytic functions.

By far the easiest case to handle is the half-plane dilation case.

Theorem

If φ is an analytic mapping of the unit disk to itself with Denjoy–Wolff point a on the unit circle and $\varphi'(a) < 1$, then for real θ the operator C_φ on $H^2(\mathbb{D})$ is similar to the operator $e^{i\theta}C_\varphi$.

Thus, if λ is in the spectrum of C_φ then for real θ, $e^{i\theta}\lambda$ is also.

(half-plane dilation)

Theorem

If φ, not an inner function, is analytic in a neighborhood of the closed unit disk, maps the disk to itself, and has Denjoy–Wolff point a on the unit circle with $\varphi'(a) < 1$, then for C_φ acting on the Hardy space $H^2(\mathbb{D})$,

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \varphi'(a)^{-1/2}\}$$

For $\varphi'(a)^{1/2} < |\lambda| < \varphi'(a)^{-1/2}$, the number λ is always an eigenvalue of infinite multiplicity for C_φ

(plane dilation)

Theorem

Let φ, not an inner function, be analytic in a neighborhood of the closed disk with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(a) = a$ for some point a with $|a| < 1$.

If C_φ is the associated composition operator on $H^2(\mathbb{D})$, then

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \rho\} \cup \left\{\varphi'(a)^k : k = 1, 2, \ldots\right\} \cup \{1\}$$

where ρ is the essential spectral radius of C_φ.

In the half-plane translation case, we have some information about the spectrum:

Theorem

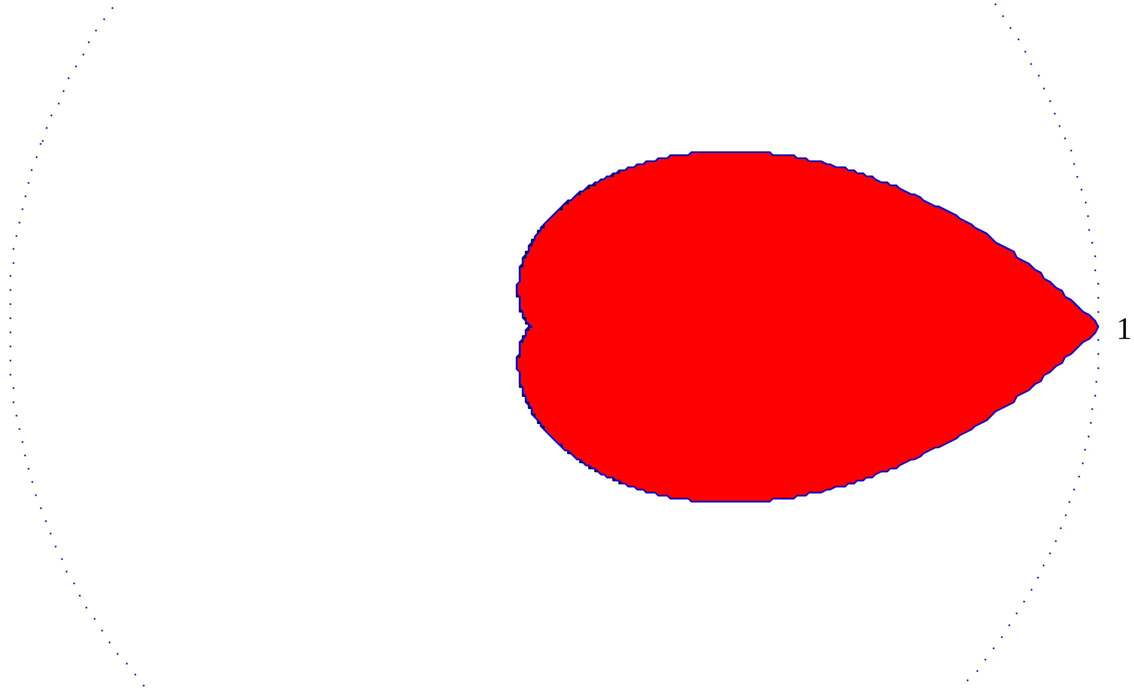
If φ is an analytic mapping of the unit disk with a halfplane/translation model for iteration, then the spectrum and essential spectrum of C_φ on $H^2(D)$ contain the unit circle. Moreover, if λ is an eigenvalue of C_φ, then $e^{i\theta}\lambda$ is also an eigenvalue of C_φ for each positive number θ.

Problem

If φ is in the half-plane translation case, not an automorphism, is $\sigma(C_\varphi)$ always $\{\lambda : |\lambda| \leq 1\}$?

In the plane translation case, we have no information about the spectrum, only a few examples:

$$\heartsuit(z) = \frac{1 + z + 2\sqrt{1 - z^2}}{3 - z + 2\sqrt{1 - z^2}}$$



We find $\sigma(C_\heartsuit)$ is the heart-shaped region $\{e^{-\beta} : |\arg\beta| \leq \pi/4\} \cup \{0\}$

In the plane translation case, the only examples for which we know the spectra are symbols that belong to a semigroup of analytic functions, and the spectrum is computed using semigroup theory.

Problem

If φ is in the plane translation case, is $\sigma(C_\varphi)$ always a union of spirals joining 0 and 1?

Problem

Find the spectrum of C_φ for a function φ in the plane translation case that is not inner, linear fractional, or a member of a semigroup of analytic functions.

Hermitian Weighted Composition Operators on $\mathcal{H}_\kappa$

Clearly, if ψ is in $H^\infty(\mathbb{D})$, then for any φ mapping the unit disk into itself, $W_{\psi,\varphi}$ is bounded on $\mathcal{H}_\kappa$ and

$$\|W_{\psi,\varphi}\| \leq \|\psi\|_\infty \|C_\varphi\|$$

BUT, it is not necessary for ψ to be bounded for $W_{\psi,\varphi}$ to be bounded!

On the other hand, since $\mathcal{H}_\kappa$ contains constants, if $W_{\psi,\varphi}$ is bounded on $\mathcal{H}_\kappa$,

then $\psi = W_{\psi,\varphi}(1)$ is in $\mathcal{H}_\kappa$.

(Write GG for: Gajath Gunatillake, Thesis, 2005)

Theorem (GG).

If $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$ *and* ψ *is in* $H^2(\mathbb{D})$*, then* $W_{\psi,\varphi}$ *is compact on* $H^2(\mathbb{D})$*.*

Theorem (GG).

Let $W_{\psi,\varphi}$ be bounded on $H^2(\mathbb{D})$ and suppose φ is continuous on the closed disk. Let $Z = \{\zeta : |\varphi(\zeta)| = 1\}$.

If, for each ζ in Z, ψ is continuous at ζ and $\psi(\zeta) = 0$, then $W_{\psi,\varphi}$ is compact on $H^2(\mathbb{D})$.

Example. Suppose $\varphi(z) = (1+z)/2$ and $\psi(z) = \left(\frac{1-z}{1+z}\right)^{1/3}$

Then $W_{\psi,\varphi}$ is compact on $H^2(\mathbb{D})$.

Note that C_φ is *not* compact and that ψ is in $H^2(\mathbb{D})$ but *not* in $H^\infty(\mathbb{D})$.

But $W_{\psi,\varphi}$ is bounded, $Z = \{1\}$, and ψ is continuous at 1 with $\psi(1) = 0$, so the compactness of $W_{\psi,\varphi}$ follows from the Theorem above.

Theorem. *Suppose* $W_{\psi,\varphi}$ *is a bounded operator on* $\mathcal{H}_\kappa$*. For* θ *a real number, let* U_θ *be the unitary composition operator given by* $(U_\theta f)(z) = f(e^{i\theta}z)$ *for* f *in* $\mathcal{H}_\kappa$*. Then*

$$U_\theta^* W_{\psi,\varphi} U_\theta = W_{\tilde{\psi},\tilde{\varphi}}$$

where $\tilde{\psi}(z) = \psi(e^{-i\theta}z)$ *and* $\tilde{\varphi}(z) = e^{i\theta}\varphi(e^{-i\theta}z)$

Corollary. *For* $W_{\psi,\varphi}$ *bounded on* $\mathcal{H}_\kappa$*, there are* η *in* $\mathcal{H}_\kappa$ *and* σ *an analytic map of the unit disk into itself with* $\sigma(0) \geq 0$ *so that the weighted composition operator* $W_{\eta,\sigma}$ *is unitarily equivalent to* $W_{\psi,\varphi}$*.*

The final part of this presentation is joint work with

Eung Il Ko of Ewha Women's University, Seoul, Korea, and

Gajath Gunatillake of Sharjah University, Sharjah, UAE.

Theorem. *If the weighted composition operator* $W_{\psi,\varphi}$ *is Hermitian on* $\mathcal{H}_\kappa$*, then* $\psi(0)$ *and* $\varphi'(0)$ *are real and* $\varphi(z) = a_0 + a_1 z/(1 - \overline{a_0} z)$

and $\psi(z) = c/(1 - \overline{a_0} z)^\kappa$

where $a_0 = \varphi(0)$, $a_1 = \varphi'(0)$, *and* $c = \psi(0)$.

Conversely, let a_0 *be in* $\mathbb{D}$, *and let* c *and* a_1 *be real numbers. If* $\varphi(z) = a_0 + a_1 z/(1 - \overline{a_0} z)$ *maps the unit disk into itself and* $\psi(z) = c/(1 - \overline{a_0} z)^\kappa$, *then the weighted composition operator* $W_{\psi,\varphi}$ *is Hermitian on* $\mathcal{H}_\kappa$.

Proof (for $\kappa = 1$): For α in the open unit disk $\mathbb{D}$, then

$$(W_{\psi,\varphi} K_\alpha)(z) = W_{\psi,\varphi}\left(\frac{1}{1-\overline{\alpha}z}\right) = \frac{\psi(z)}{1-\overline{\alpha}\varphi(z)}$$

On the other hand,

$$W^*_{\psi,\varphi}(K_\alpha)(z) = \overline{\psi(\alpha)} K_{\varphi(\alpha)}(z) = \frac{\overline{\psi(\alpha)}}{1-\overline{\varphi(\alpha)}z}$$

Thus, $W_{\psi,\varphi}$ is Hermitian if and only if

$$\frac{\psi(z)}{1-\overline{\alpha}\varphi(z)} = \frac{\overline{\psi(\alpha)}}{1-\overline{\varphi(\alpha)}z}$$

for all α and z in the unit disk.

Setting $\alpha = 0$

$$\psi(z) = \frac{\overline{\psi(0)}}{1-\overline{\varphi(0)}z}$$

for all z in the disk. Setting $z = 0$, we get $\psi(0) = \overline{\psi(0)}$, so that $\psi(0)$ is real.

Defining c and a_0 by $c = \psi(0)$ and $a_0 = \varphi(0)$, we can write ψ as

$$\psi(z) = \frac{c}{1 - \overline{a_0} z}$$

Now, recalling

$$\frac{\psi(z)}{1 - \overline{\alpha}\varphi(z)} = \frac{\overline{\psi(\alpha)}}{1 - \overline{\varphi(\alpha)} z}$$

by substituting for ψ and representing φ as a series and collecting terms, we get

$$\varphi(z) = a_0 + a_1 z / (1 - \overline{a_0} z)$$

where $a_1 = \varphi'(0)$.

The converse follows by reversing the calculations.

Proposition. *Let a_1 be real. Then $\varphi(z) = a_0 + a_1 z/(1 - \overline{a_0} z)$ maps the open unit disk into itself if and only if*

$$|a_0| < 1 \quad \textit{and} \quad -1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2$$

The three cases,

$$a_1 = -1 + |a_0|^2,$$

$$-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2,$$

and

$$a_1 = (1 - |a_0|)^2$$

are quite different from each other.

Case 1:

When $|a_0| < 1$ and $a_1 = -1 + |a_0|^2$, then

$$\varphi(z) = a_0 + \frac{(-1 + |a_0|^2)z}{1 - \overline{a_0}z} = \frac{z - a_0}{\overline{a_0}z - 1}$$

is an automorphism with $\varphi(\varphi(z)) = z$.

Choosing $c = \pm(1 - |a_0|^2)^{-\kappa/2}$, we get $W_{\psi,\varphi}$ as a Forelli-like isometry and

$$W_{\psi,\varphi}^2 = I.$$

Taking $a_0 = 0$ gives $W_{\psi,\varphi} = \pm I$.

When $a_0 \neq 0$, we have

$$\text{spectrum } W_{\psi,\varphi} = \{-1, 1\}$$

and if b is the fixed point of φ in the open unit disk, then the set

$$e_j(z) = \frac{(1 - |b|^2)^{\kappa/2}}{(1 - \bar{b}z)^{\kappa}} \left(\frac{z - b}{\bar{b}z - 1} \right)^j$$

is an orthonormal basis for $\mathcal{H}_\kappa$ consisting of eigenvectors for $W_{\psi,\varphi}$, where the eigenspace corresponding to the eigenvalue 1 for $W_{\psi,\varphi}$ is the closed span of $\{e_j : j = 0, 2, 4, \cdots\}$, and the eigenspace corresponding to the eigenvalue -1 is the closed span of $\{e_j : j = 1, 3, 5, \cdots\}$.

Case 2:

When $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$,

φ maps the closed disk into the open disk and $W_{\psi,\varphi}$ is compact.

If b is the fixed point of φ in the open unit disk, the spectrum of $W_{\psi,\varphi}$ is

$$\{\psi(b), \psi(b)\varphi'(b), \psi(b)\varphi'(b)^2, \psi(b)\varphi'(b)^3, \cdots\} \cup \{0\}$$

and the set

$$e_j(z) = \frac{(1 - |b|^2)^{\kappa/2}}{(1 - \overline{b}z)^{\kappa}} \left(\frac{z - b}{\overline{b}z - 1}\right)^j$$

is an orthonormal basis for $\mathcal{H}_\kappa$ consisting of eigenvectors for $W_{\psi,\varphi}$.

Case 3:

When $a_1 = (1 - |a_0|)^2$, $a_0 \neq 0$,

the map φ has a fixed point on the unit circle, (none in the disk),

but is not an automorphism of the disk, and $W_{\psi,\varphi}$ is not compact.

By normalizing, we may assume $0 < a_0 < 1$.

Writing $t = a_0/(1 - a_0)$, each such $W_{\psi,\varphi}$ is a multiple of $A_t = W_{\psi_t,\varphi_t}$ where

$$\psi_t = (1 + t - tz)^{-\kappa}$$

and

$$\varphi_t = (t + (1 - t)z)/(1 + t - tz)$$

Then for $0 \leq t < \infty$, A_t is a semigroup of Hermitian weighted composition operators. (And(!!) for $\operatorname{Re} t > 0$, A_t is a semigroup of normal operators.)

Theorem.

For $\kappa \geq 1$ and $0 \leq t < \infty$, let $A_t = W_{\psi_t,\varphi_t}$ where

$$\psi_t = (1 + t - tz)^{-\kappa} \quad \textit{and} \quad \varphi_t = (t + (1-t)z)/(1 + t - tz)$$

The A_t form a strongly continuous semigroup of Hermitian weighted composition operators on $\mathcal{H}_\kappa$. If Δ is the infinitesimal generator of this semigroup, $\mathcal{D}_A = \{f \in \mathcal{H}_\kappa : (z-1)^2 f' \in \mathcal{H}_\kappa\}$ is the domain of Δ and $\Delta(f)(z) = (z-1)^2 f'(z) + \kappa(z-1)f(z) \quad$ for f in $\mathcal{D}_A$.

Corollary.

For $\kappa \geq 1$ and for $t > 0$, the operator A_t on $\mathcal{H}_\kappa$ has no eigenvalues.

Proof: There are no non-zero functions in $\mathcal{H}_\kappa$ that satisfy

$$(z-1)^2 f' + \kappa(z-1)f = \lambda f(z)$$

Theorem. (Spectral Theorem, version 1)

For $\kappa \geq 1$ and $0 \leq t < \infty$, let $A_t = W_{\psi_t,\varphi_t}$ where

$$\psi_t = (1 + t - tz)^{-\kappa} \quad \text{and} \quad \varphi_t = (t + (1-t)z)/(1 + t - tz)$$

For each t, the operator A_t is a cyclic Hermitian weighted composition operator on $\mathcal{H}_\kappa$. Indeed, the vector 1 is a cyclic vector for A_t.

If μ is the absolutely continuous probability measure given by

$$d\mu = \frac{(\ln(1/x))^{\kappa-1}}{\Gamma(\kappa)}\, dx$$

the operator U given by $U(\psi_t) = x^t$ for $0 \leq t < \infty$, is a unitary map of $\mathcal{H}_\kappa$ onto $L^2([0,1],\mu)$ and satisfies $UA_t = M_{x^t}U$.

In particular, for each $t > 0$, these operators satisfy $\|A_t\| = 1$ and have spectrum $\sigma(A_t) = [0,1]$.

We define subspaces H_c of $\mathcal{H}_\kappa = A^2_{\kappa-2}$ as follows:

Let $H_0 = \mathcal{H}_\kappa$. For $c < 0$, define the subspace H_c by

$$H_c = \text{ closure } \{e^{c\frac{1+z}{1-z}} f : f \in \mathcal{H}_\kappa\}$$

For $0 \leq t$ and $c \leq 0$, the subspace H_c is invariant for A_t.

For $0 \leq \delta \leq 1$ define the subspace L_δ of $L^2([0,1],\mu)$ by

$$L_\delta = \{f \in L^2([0,1],\mu) : f(x) = 0 \text{ for } \delta < x \leq 1\}$$

These are spectral subspaces of the multiplication operators M_{x^t}

Theorem. (Spectral Theorem, version 2)

If U gives unitary equivalence from A_t on $\mathcal{H}_\kappa$ to M_{x^t} on $L^2([0,1],\mu)$,

$$\textit{then} \quad U^* L_\delta = H_{(\ln \delta)/2} \quad \textit{or equivalently} \quad U H_c = L_{e^{2c}}$$

Suppose N is a subspace of $\mathcal{H}_\kappa$ that is invariant for the operator of multiplication by z.

If there is f in N with $f(0) \neq 0$ and G is a function of N so that

$$\|G\| = 1 \quad \text{and} \quad G(0) = \sup\{\operatorname{Re} f(0) : f \in N \text{ and } \|f\| = 1\}$$

then we say G *solves the extremal problem for the invariant subspace N.*

Subspaces H_c are spectral subspaces for A_t, but more interestingly, they are invariant subspaces for M_z on $\mathcal{H}_\kappa$ generated by atomic inner functions!

The unitary equivalence between the subspaces H_c in $\mathcal{H}_\kappa$ and L_δ in $L^2([0,1],\mu)$ gives an opportunity to compute the extremal functions for L_δ and translate the answer back to H_c!!

Our computation requires the use of the *incomplete Gamma function*

$$\Gamma(a, w) = \int_w^\infty t^{a-1} e^{-t} \, dt$$

where a is a complex parameter and w is a real parameter. An alternate definition in which both a and w are complex parameters is

$$\Gamma(a, w) = e^{-w} w^a \int_0^\infty e^{-wu} (1+u)^{a-1} \, du$$

Theorem.

For $c < 0$, if H_c is the invariant subspace of $\mathcal{H}_\kappa$ defined by

$$H_c = \text{closure}\{e^{c\frac{1+z}{1-z}} f : f \in \mathcal{H}_\kappa\}$$

then the extremal function for H_c is

$$G_c(z) = \frac{\Gamma(\kappa, -2c/(1-z))}{\sqrt{\Gamma(\kappa)}\sqrt{\Gamma(\kappa, -2c)}}$$

Theorem.

For $0 < r < 1$, let P_r be the orthogonal projection onto the subspace $H_{(\ln r)/2}$ in $\mathcal{H}_\kappa$. If u is any point of the open unit disk, then for $K_u(z) = (1 - \overline{u}z)^{-\kappa}$

$$(P_r K_u)(z) = \frac{1}{\Gamma(\kappa)(1-\overline{u}z)^\kappa}\Gamma\left(\kappa, -\frac{(\ln r)(1-\overline{u}z)}{(1-\overline{u})(1-z)}\right)$$

This gives the kernel functions for the invariant subspaces H_c in $\mathcal{H}_\kappa$, including for the usual Bergman space ($\kappa = 2$). This result generalizes the formula for the usual Bergman space computed in a different way by W. Yang in his thesis.

THANK YOU!!

http://www.math.iupui.edu/~ccowen/Downloads.html

Invariant Subspaces: A complete lattice!

Theorem (Montes-Rodríguez, Ponce-Escudero, & Shkarin, '10)

For $\operatorname{Re} a > 0$, let

$$\varphi_a(z) = \frac{(2-a)z + a}{-az + 2 + a}$$

A closed subspace M of $H^2(\mathbb{D})$ is invariant for C_{φ_a} if and only if there is a closed set F of $[0, \infty)$ such that

$$M = \text{closed span}\{e^{t\frac{z+1}{z-1}} : t \in F\}$$

The relevance of the functions $e^{t\frac{z+1}{z-1}}$ is that they are eigenvectors for C_{φ_a}:

$$C_{\varphi_a}\left(e^{t\frac{z+1}{z-1}}\right) = e^{-at}e^{t\frac{z+1}{z-1}}$$

In other words, each of the invariant subspaces for C_{φ_a} is the closed span of a collection of eigenvectors.

Theorem (Montes-Rodríguez, Ponce-Escudero, & Shkarin, 2010)

For $\operatorname{Re} a > 0$, let

$$\varphi_a(z) = \frac{(2-a)z + a}{-az + 2 + a}$$

A closed subspace M of $H^2(\mathbb{D})$ is invariant for C_{φ_a} if and only if there is a closed set F of $[0, \infty)$ such that

$$M = \text{closed span}\{e^{t\frac{z+1}{z-1}} : t \in F\}$$

Corollary

If $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$,

then the lattices of invariant subspaces for C_{φ_a} and for C_{φ_b} are the same.

Corollary

If $\operatorname{Re} a > 0$, then C_{φ_a} has no (non-trivial) reducing subspaces.

Rota's Universal Operators:

Defn: Let $\mathcal{X}$ be a Banach space, let U be a bounded operator on $\mathcal{X}$, and let $\mathcal{B}(\mathcal{X})$ be the algebra of bounded operators on $\mathcal{X}$.

We say U is *universal for* $\mathcal{X}$ if for each non-zero bounded operator A on $\mathcal{X}$, there is an invariant subspace M for U and a non-zero number λ such that λA is similar to $U|_M$.

Rota proved in 1960 that if $\mathcal{X}$ is a separable, infinite dimensional Hilbert space, there are universal operators on $\mathcal{X}$!

Theorem.

The Toeplitz operator T_φ^* is universal for $H^2(\mathbb{D})$.

Main Theorem.

The operator $W_{\psi,J}^*$ is an injective, compact operator that commutes with the universal operator T_φ^*.