Invariant Subspaces for Composition Operators

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If φ is an analytic function of $\mathbb{D}$ into itself,

and $\mathcal{H}$ is a Hilbert space of analytic functions on $\mathbb{D}$,

then the composition operator C_φ on $\mathcal{H}$ is the operator

$$C_\varphi f = f \circ \varphi \quad \text{for} \quad f \in \mathcal{H}$$

Usual spaces: f analytic in $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\text{Hardy:} \quad H^2(\mathbb{D}) = H^2 = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

$$\text{Bergman:} \quad A^2(\mathbb{D}) = A^2 = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Bergman } (\alpha > 0): \quad A^2_\alpha = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Hardy } (\|z^n\| = \beta_n > 0): \quad H^2(\beta) = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty\}$$

Recall: For w in $\mathbb{D}$, the *reproducing kernel function for* $\mathcal{H}$ is K_w in $\mathcal{H}$ with

$$\langle f, K_w \rangle = f(w) \quad \text{for all} \quad f \in \mathcal{H}$$

For H^2, we have $K_w(z) = (1 - \overline{w}z)^{-1}$

For A^2, we have $K_w(z) = (1 - \overline{w}z)^{-2}$

In this talk, we will consider spaces $H^2(\beta_\kappa)$ for $\kappa \geq 1$ which are the weighted Hardy spaces with

$$K_w(z) = (1 - \overline{w}z)^{-\kappa}$$

The spaces $H^2(\beta_\kappa)$ include the usual Hardy and Bergman spaces and all the weighted Bergman spaces ($\alpha = \kappa + 2$).

On all of these spaces, for any φ analytic map of $\mathbb{D}$ into itself, the composition operator C_φ is a bounded operator and for all w in $\mathbb{D}$

$$C_\varphi^* K_w = K_{\varphi(w)}$$

For A a bounded operator on $\mathcal{H}$, a (closed) subspace M is called a (non-trivial) *invariant subspace of* A if $M \neq 0$ and $M \neq \mathcal{H}$ and

$$v \in M \quad \text{implies} \quad Av \in M \quad \text{also.}$$

In finite dimensional spaces, every operator has invariant subspaces

and understanding the structure of the invariant subspaces

has been critical in understanding the structure of the operators.

Want the same for operators on infinite dimensional spaces!

Invariant Subspace Problem:

Does every bounded operator have a (non-trivial) invariant subspace?

No! in general, for Banach spaces! (C. J. Read and others 1984–)

Still open for Hilbert spaces!

BUT,

for Hilbert space operator for which the lattice of invariant subspaces is known,

we feel we have a basic understanding of the structure of the operator!

Goal today:

Outline three sets of ideas about invariant subspaces of composition operators

and thereby persuade you that now is a good time to think about this topic!

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A Digression!

One of the first times that invariant subspaces were mentioned in connection with composition operators was in a paper [NRW] of Nordgren, Rosenthal, and Wintrobe: "Invertible Composition Operators on H^p"

J. Func. Anal. **73**(1987), 324–344.

Definition:

An operator U is called *universal* if for every operator T, some multiple of T is similar to the restriction of U to one of its invariant subspaces.

Caradus (1969) showed that

An operator is universal if it is onto and has an infinite dimensional kernel

and [NRW] showed that

For $\varphi(z) = \dfrac{2z-1}{2-z}$ the operator $C_\varphi - I$ is universal.

End of Digression!

BUT,

for Hilbert space operator for which the lattice of invariant subspaces is known,

we feel we have a basic understanding of the structure of the operator!

Beurling's Theorem (1949):

Let S be the operator of multiplication by z on $H^2(\mathbb{D})$. A closed subspace M of $H^2(\mathbb{D})$ is invariant for S if and only if there is an inner function ψ such that $M = \psi H^2(\mathbb{D})$.

First Example: A complete lattice!

Theorem (Montes-Rodríguez, Ponce-Escudero, & Shkarin, 2010)

For $\mathrm{Re}\, a > 0$, let

$$\varphi_a(z) = \frac{(2-a)z + a}{-az + 2 + a}$$

A closed subspace M of $H^2(\mathbb{D})$ is invariant for C_{φ_a} if and only if there is a closed set F of $[0, \infty)$ such that

$$M = \text{closed span}\{e^{t\frac{z+1}{z-1}} : t \in F\}$$

The relevance of the functions $e^{t\frac{z+1}{z-1}}$ is that they are eigenvectors for C_{φ_a}:

$$C_{\varphi_a}\left(e^{t\frac{z+1}{z-1}}\right) = e^{-at}e^{t\frac{z+1}{z-1}}$$

In other words, each of the invariant subspaces for C_{φ_a} is the closed span of a collection of eigenvectors.

Theorem (Montes-Rodríguez, Ponce-Escudero, & Shkarin, 2010)

For $\operatorname{Re} a > 0$, let

$$\varphi_a(z) = \frac{(2-a)z + a}{-az + 2 + a}$$

A closed subspace M of $H^2(\mathbb{D})$ is invariant for C_{φ_a} if and only if there is a closed set F of $[0, \infty)$ such that

$$M = \text{closed span}\{e^{t\frac{z+1}{z-1}} : t \in F\}$$

Corollary

If $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$,

then the lattices of invariant subspaces for C_{φ_a} and for C_{φ_b} are the same.

Corollary

If $\operatorname{Re} a > 0$, then C_{φ_a} has no (non-trivial) reducing subspaces.

Their proof is based on two quite different ideas.

First, suppose $\mathcal{A}$ is a Banach algebra. If τ is in $\mathcal{A}$, we say τ is *a cyclic element* if the algebra generated by τ is dense in $\mathcal{A}$.

Let M_τ be the operator on $\mathcal{A}$ of multiplication by τ, that is,

$$M_\tau \omega = \tau\omega \text{ for } \omega \text{ in } \mathcal{A}.$$

Proposition

If τ is a cyclic element in the Banach algebra $\mathcal{A}$,

then the invariant subspaces of M_τ are the closed ideals of $\mathcal{A}$.

Let $W^{1,2}[0,\infty)$ be the Sobolev space with inner product

$$\langle f, g\rangle_{1,2} = \frac{1}{2}\int_0^\infty f(t)\overline{g(t)} + f'(t)\overline{g'(t)}\,dt$$

where f and g are functions in $L^2[0,\infty)$ that are absolutely continuous on each bounded subinterval of $[0,\infty)$ and whose derivatives f' and g' are in $L^2[0,\infty)$.

They give a unitary equivalence between $H^2(\mathbb{D})$ and the Sobolev space and they show that $W^{1,2}[0,\infty)$ is a Banach algebra.

Finally, they show that the unitary equivalence of these spaces carries the adjoints of the composition operators to multiplication by cyclic elements of the Banach algebra to which they can apply the Proposition. ∎

Second Example:

Invariant subspaces with application to function theory

Let φ be an analytic map of $\mathbb{D}$ into itself and let ψ be analytic on $\mathbb{D}$.

The *weighted composition operator* $W_{\psi,\varphi}$ is the operator on $H^2(\beta_\kappa)$ given by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z)) \quad \text{for } z \text{ in } \mathbb{D}$$

Since $H^2(\beta_\kappa)$ contains the constants,

if $W_{\psi,\varphi}$ is bounded, then $\psi = W_{\psi,\varphi}(1)$ is in $H^2(\beta_\kappa)$.

Clearly, if ψ is in $H^\infty(\mathbb{D})$ then $W_{\psi,\varphi}$ is a bounded operator on $H^2(\beta_\kappa)$ and

$$\|W_{\psi,\varphi}\| \leq \|\psi\|_\infty \|C_\varphi\|$$

BUT, it is not necessary for ψ to be bounded for $W_{\psi,\varphi}$ to be bounded.

Theorem. (Ko & C. for $H^2(\mathbb{D})$ and Gunatillake, Ko, and C. for $H^2(\beta_\kappa)$)

For $\kappa \geq 1$,

$W_{\psi,\varphi}$ is a bounded Hermitian weighted composition operator if and only if

$$\psi(z) = c(1 - \overline{a_0}z)^{-\kappa} \qquad \textit{and } \varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a_0}z}$$

where $c = \psi(0)$ and $a_1 = \varphi'(0)$ are real numbers

and a_1 and $a_0 = \varphi(0)$ are such that φ maps the unit disk into itself.

Without loss of generality, $0 < a_0 < 1$, and then the most interesting case comes when $a_1 = (1 - a_0)^2$ which means that $\varphi(1) = \varphi'(1) = 1$.

Writing $t = a_0/(1 - a_0)$, each such $W_{\psi,\varphi}$ is a multiple of $A_t = W_{\psi_t,\varphi_t}$ where

$$\psi_t(z) = (1 + t - tz)^{-\kappa} \quad \text{and} \quad \varphi_t(z) = (t + (1 - t)z)/(1 + t - tz)$$

Theorem.

For $\kappa \geq 1$ *and* $0 \leq t < \infty$, *let* $A_t = W_{\psi_t,\varphi_t}$ *where*

$$\psi_t(z) = (1 + t - tz)^{-\kappa} \quad \text{and} \quad \varphi_t(z) = (t + (1-t)z)/(1 + t - tz)$$

The A_t *form a strongly continuous semigroup of Hermitian weighted composition operators on* $H^2(\beta_\kappa)$. *If* Δ *is the infinitesimal generator of this semigroup,* $\mathcal{D}_A = \{f \in H^2(\beta_\kappa) : (z-1)^2 f' \in H^2(\beta_\kappa)\}$ *is the domain of* Δ *and* $(\Delta f)(z) = (z-1)^2 f'(z) + \kappa(z-1)f(z)$ *for* f *in* $\mathcal{D}_A$.

Corollary.

For $\kappa \geq 1$ *and for* $t > 0$, *the operator* A_t *on* $H^2(\beta_\kappa)$ *has no eigenvalues.*

Proof: There are no non-zero functions in $H^2(\beta_\kappa)$ that satisfy

$$(z-1)^2 f' + \kappa(z-1)f = \lambda f$$

Theorem.

For $\kappa \geq 1$ *and* $0 \leq t < \infty$, *let* $A_t = W_{\psi_t,\varphi_t}$ *where*

$$\psi_t(z) = (1 + t - tz)^{-\kappa} \quad \text{and} \quad \varphi_t(z) = (t + (1-t)z)/(1 + t - tz)$$

For each $t > 0$, *the operator* A_t *is a cyclic Hermitian weighted composition operator on* $H^2(\beta_\kappa)$. *Indeed, the vector* 1 *is a cyclic vector for* A_t.

If μ *is the absolutely continuous probability measure given by*

$$d\mu = \frac{(\log(1/x))^{\kappa-1}}{\Gamma(\kappa)}\, dx$$

the operator U *given by* $U(\psi_t) = x^t$ *for* $0 \leq t < \infty$ *is a unitary map of* $H^2(\beta_\kappa)$ *onto* $L^2([0,1],\mu)$ *and satisfies* $UA_t = M_{x^t}U$.

In particular, for each $t > 0$, *these operators satisfy* $\|A_t\| = 1$ *and have spectrum* $\sigma(A_t) = [0,1]$.

We define subspaces H_c of $H^2(\beta_\kappa) = A^2_{\kappa-2}$ as follows:

Let $H_0 = H^2(\beta_\kappa)$. For $c < 0$, define the subspace H_c by

$$H_c = \text{closure}\{e^{c\frac{1+z}{1-z}} f : f \in H^2(\beta_\kappa)\}$$

For $0 \leq t$ and $c \leq 0$, the subspace H_c is invariant for A_t.

For $0 \leq \delta \leq 1$ define the subspace L_δ of $L^2([0,1],\mu)$ by

$$L_\delta = \{f \in L^2([0,1],\mu) : f(x) = 0 \;\text{ for }\; \delta < x \leq 1\}$$

These are the spectral subspaces of the multiplication operators M_{x^t}.

Theorem.

If U gives the unitary equavalence from A_t on $H^2(\beta_\kappa)$ to M_{x^t} on $L^2([0,1],\mu)$,

then $\quad U^* L_\delta = H_{(\log \delta)/2} \quad$ *or equivalently* $\quad U H_c = L_{e^{2c}}$

Suppose N is a subspace of $H^2(\beta_\kappa)$ that is invariant for the operator of multiplication by z.

If there is f in N with $f(0) \neq 0$ and G is a function of N so that

$$\|G\| = 1 \quad \text{and} \quad G(0) = \sup\{\operatorname{Re} f(0) : f \in N \ \text{ and } \ \|f\| = 1\}$$

then we say G *solves the extremal problem for the invariant subspace* N.

Subspaces H_c are spectral subspaces for A_t, but more interestingly, they are invariant subspaces for M_z on $H^2(\beta_\kappa)$ generated by atomic inner functions!

The unitary equivalence between the subspaces H_c in $H^2(\beta_\kappa)$ and L_δ in $L^2([0,1],\mu)$ gives an opportunity to compute the extremal functions for L_δ and translate the answer back to H_c!!

Our computation requires the use of the *incomplete Gamma function*

$$\Gamma(a, w) = \int_w^\infty t^{a-1} e^{-t}\, dt$$

where a is a complex parameter and w is a real parameter. An alternate definition in which both a and w are complex parameters is

$$\Gamma(a, w) = e^{-w} w^a \int_0^\infty e^{-wu} (1+u)^{a-1}\, du$$

Theorem.

For $c < 0$, if H_c is the invariant subspace for M_z in $H^2(\beta_\kappa)$ defined by

$$H_c = \text{closure}\{e^{c\frac{1+z}{1-z}} f : f \in H^2(\beta_\kappa)\}$$

then the extremal function for H_c is

$$G_c(z) = \frac{\Gamma(\kappa, -2c/(1-z))}{\Gamma(\kappa)\Gamma(\kappa, -2c)}$$

Theorem.

For $0 < r < 1$*, let* P_r *be the orthogonal projection onto the subspace* $H_{(\log r)/2}$ *in* $H^2(\beta_\kappa)$*. If* u *is any point of the open disk, then for* $K_u(z) = (1 - \overline{u}z)^{-\kappa}$

$$(P_r K_u)(z) = \frac{1}{\Gamma(\kappa)(1 - \overline{u}z)^{\kappa}} \Gamma\left(\kappa, -\frac{(\log r)(1 - \overline{u}z)}{(1 - \overline{u})(1 - z)}\right)$$

This gives the kernel functions for the invariant subspaces H_c in $H^2(\beta_\kappa)$, including for the usual Bergman space ($\kappa = 2$).

This result generalizes the formula for the usual Bergman space computed in a different way by W. Yang in his thesis.

Third Example:

Common invariant subspaces for C_φ and S, multiplication by z

(Wahl & C., 2011)

Let φ be an analytic map of $\mathbb{D}$ into itself.

We say b *is a fixed point of* φ if $\varphi(b) = b$ (when $|b| < 1$),

or if $\lim_{r\to 1^-} \varphi(rb) = b$ (when $|b| = 1$).

Julia-Carathéodory Theorem implies

If b is a fixed point of φ with $|b| = 1$, then $\lim_{r\to 1^-} \varphi'(rb)$ exists

(call it $\varphi'(b)$) and $0 < \varphi'(b) \leq \infty$.

Denjoy-Wolff Theorem (1926).

If φ is an analytic map of $\mathbb{D}$ into itself (not an elliptic automorphism), there is a unique fixed point, a, of φ in $\overline{\mathbb{D}}$ such that $|\varphi'(a)| \leq 1$.

Moreover, the sequence of iterates, (φ_n), converges to a uniformly on compact subsets of $\mathbb{D}$, and this distinguished fixed point is called the *Denjoy-Wolff point*.

Analytic self-maps (not elliptic automorphisms) of $\mathbb{D}$ divide into distinct classes based on linear fractional models for iteration:

- (Plane/Dilation): $|a| < 1$ and $0 < |\varphi'(a)| < 1$
- (Half-Plane/Dilation): $|a| = 1$ and $0 < \varphi'(a) < 1$
- (Half-Plane/Translation): $|a| = 1$ and $\varphi'(a) = 1$, and $\{\varphi_n(z)\}$ interpolating
- (Plane/Translation): $|a| = 1$ and $\varphi'(a) = 1$, and $\{\varphi_n(z)\}$ not interpolating
- (no LF model): $|a| < 1$ and $\varphi'(a) = 0$

Always assume that φ is non-constant and not an elliptic automorphism.

Without loss of generality, if a, the Denjoy-Wolff point of φ, is in $\mathbb{D}$, we can assume $a = 0$ and if $|a| = 1$, we can assume $a = 1$.

For simplicity, we will assume that the Hilbert space is $H^2(\mathbb{D})$, although many of the results hold for $H^2(\beta_\kappa)$. When weighted composition operators, $W_{\psi,\varphi}$, are discussed, we will assume that ψ is in $H^\infty(\mathbb{D})$.

Theorem.

If φ is an analytic map of $\mathbb{D}$ into itself, ψ is in H^∞, and M is an invariant subspace for C_φ and for S, then M is an invariant subspace for $W_{\psi,\varphi}$.

Conversely, if ψ^{-1} is in H^∞ and M is an invariant subspace for $W_{\psi,\varphi}$ and for S, then M is invariant for C_φ.

Theorem.

If φ is an analytic map of the unit disk into itself with $\varphi(1) = 1$ and $\varphi'(1) \leq 1$, then $e^{\alpha\frac{z+1}{z-1}}H^2$ is an invariant subspace for C_φ when $\alpha > 0$.

Outline of Proof:

Use Julia's Lemma to prove the following:

Let φ be an analytic map of the unit disk into itself with $\varphi(1) = 1$ and $\varphi'(1) \leq 1$. Then for z in $\mathbb{D}$, $\text{Re}\left(\frac{\varphi(z)+1}{\varphi(z)-1} - \frac{z+1}{z-1}\right) \leq 0$

For g in H^2,

$$\begin{aligned} C_\varphi(e^{\alpha\frac{z+1}{z-1}}g)(z) &= e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}g(\varphi(z)) \\ &= \left(e^{\alpha\frac{z+1}{z-1}}e^{\alpha\left(\frac{\varphi(z)+1}{\varphi(z)-1}-\frac{z+1}{z-1}\right)}\right)g(\varphi(z)) \\ &= e^{\alpha\frac{z+1}{z-1}}\left(e^{\alpha\left(\frac{\varphi(z)+1}{\varphi(z)-1}-\frac{z+1}{z-1}\right)}g(\varphi(z))\right) \end{aligned}$$

∎

Theorem.

If φ is an analytic map of the unit disk into itself with $\varphi(1) = 1$ and $\varphi'(1) \leq 1$, then $e^{\alpha\frac{z+1}{z-1}}H^2$ is an invariant subspace for C_φ when $\alpha > 0$.

Conversely, if φ is an analytic map of the disk into itself and $e^{\alpha\frac{z+1}{z-1}}H^2$ is an invariant subspace for C_φ for some $\alpha > 0$, then $\varphi(1) = 1$ and $\varphi'(1) \leq 1$.

Theorem.

Conversely, if φ *is an analytic map of the disk into itself and* $e^{\alpha\frac{z+1}{z-1}}H^2$ *is an invariant subspace for* C_φ *for some* $\alpha > 0$*, then* $\varphi(1) = 1$ *and* $\varphi'(1) \leq 1$*.*

Outline of Proof of Converse:

Let $M = e^{\alpha\frac{z+1}{z-1}}H^2$. Now, $e^{\alpha\frac{z+1}{z-1}}$ in M implies $e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}$ is in M also. This means $\lim_{r\to 1^-} e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}} = 0$ so that $\varphi(1) = 1$.

Using the Julia-Carathéodory Theorem, we see $e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}$ in $e^{\alpha\frac{z+1}{z-1}}H^2$ implies $\varphi'(1) \leq 1$. ∎

Conclusion:

Suppose φ *maps the disk into itself and* $\alpha > 0$*. Then the subspace* $e^{\alpha\frac{z+1}{z-1}}H^2$ *is invariant for* C_φ *if and only if* 1 *is the Denjoy-Wolff point of* φ*.*

For $|\lambda| = 1$ and z_j for $j = 1, 2, \cdots$, points in $\mathbb{D}$ satisfying $\sum_j (1 - |z_j|) < \infty$, the function

$$B(z) = \lambda \prod_j \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j} z}$$

is a Blaschke product. The zero set, $\{z_j\}$, for B is denoted $Z(B)$.

Lemma.

Let C_φ be a composition operator on H^2. Then BH^2 is invariant for C_φ if and only if z_j in $Z(B)$ implies $\varphi_n(z_j)$ is in $Z(B)$ for all non-negative integers j and n and if w is in $Z(B)$, then multiplicity $\varphi(w) \geq$ multiplicity w.

Outline of Proof.

BH^2 invariant for C_φ ensures that there is g in H^2 so that $C_\varphi B = Bg$. Thus, $C_\varphi B = 0$ whenever $B = 0$, so z_j in $Z(B)$ implies $0 = (C_\varphi B)(z_j) = B(\varphi(z_j))$, that is, $\varphi(z_j)$ is in $Z(B)$ also. ∎

Theorem.

Suppose C_φ is a composition operator on H^2 with $\varphi(a) = a$ for a in $\mathbb{D}$.

If BH^2 is a non-trivial invariant subspace for C_φ, then

(i) a is in $Z(B)$

and (ii) for every z_j in $Z(B)$, there is an integer n_j so that $\varphi(z_{n_j}) = a$.

Outline of Proof.

If w is in $Z(B)$, then, by the Lemma, $\varphi_k(w)$ is in $Z(B)$ for all k, but $\lim_{k\to\infty} \varphi_k(w) = a$. If there were infinitely many points $\varphi_k(w)$, then $B \equiv 0$, which is not the case, so there are only finitely many and there is n so that $\varphi_n(w) = a$. This means a is in $Z(B)$. ∎

Theorem.

Suppose C_φ *is a composition operator on* H^2 *with* $\varphi(a) = a$ *for* a *in* $\mathbb{D}$.

If BH^2 *is a non-trivial invariant subspace for* C_φ*, then*

(i) a *is in* $Z(B)$

and (ii) for every z_j *in* $Z(B)$*, there is an integer* n_j *so that* $\varphi(z_{n_j}) = a$.

Corollary.

Let φ *be a univalent analytic function mapping the disk into itself with* $\varphi(a) = a$ *for some* a *in* $\mathbb{D}$.

Then the subspaces $\left(\frac{z-a}{1-\overline{a}z}\right)^k H^2$ *are the only non-trivial Blaschke-product induced subspaces invariant for both* C_φ *and* S.

Corollary.

Let φ be a univalent analytic function mapping the disk into itself with $\varphi(a) = a$ for some a in $\mathbb{D}$.

Then the subspaces $\left(\frac{z-a}{1-\overline{a}z}\right)^k H^2$ are the only non-trivial Blaschke-product induced subspaces invariant for both C_φ and S.

Outline of Proof.

If B were a Blaschke product with a zero $w \neq a$ with BH^2 invariant for C_φ, then $\varphi_n(w) = a$ for some n. Because φ is univalent, φ_n is univalent for each n. But, because $\varphi_n(a) = a$ for each n, we see $\varphi_n(w) \neq a$ for any n.

This contradiction means that the only zero of B is a. ■

Theorem.

Let φ be an analytic function mapping the disk into itself with Denjoy-Wolff point on the unit circle.

If B is a Blaschke product and BH^2 is invariant for C_φ, then for each w in $Z(B)$, the set $\{\varphi_n(w) : n \in \mathbb{N}\}$ is an infinite set in $Z(B)$.

Proof.

The function φ has no fixed points in $\mathbb{D}$, so each of the points $\varphi_n(w)$ are distinct, for n in $\mathbb{N}$.

∎

Theorem.

Let φ be an analytic function mapping the disk into itself with Denjoy-Wolff point on the unit circle.

If B is a Blaschke product and BH^2 is invariant for C_φ, then for each w in $Z(B)$, the set $\{\varphi_n(w) : n \in \mathbb{N}\}$ is an infinite set in $Z(B)$.

Corollary.

Let φ be an analytic function mapping the disk into itself with Denjoy-Wolff point on the unit circle.

Then there are no finite Blaschke products B so that BH^2 is a (non-trivial) invariant subspace for C_φ. ■

Theorem.

Let φ be an analytic function mapping the disk into itself with Denjoy-Wolff point 1 *on the unit circle.*

If B is a Blaschke product and BH^2 is invariant for C_φ, then for each w in $Z(B)$, the set $\{\varphi_n(w) : n \in \mathbb{N}\}$ is an infinite set in $Z(B)$.

Corollary.

Let φ be an analytic function mapping the disk into itself with Denjoy-Wolff point $1 = \varphi(1)$ on the unit circle and suppose φ is in the half-plane translation or half-plane dilation case.

Then if B_0 is any finite Blaschke product, there is a Blaschke product B so that B_0 divides B and BH^2 is an invariant subspace for C_φ.

Outline of Proof.

In these cases, if w is a zero of B_0, then $\{\varphi_n(w)\}$ is a Blaschke sequence. ∎

Theorem.

Let φ be an analytic function mapping the disk into itself with Denjoy-Wolff point 1 on the unit circle.

If B is a Blaschke product and BH^2 is invariant for C_φ, then for each w in $Z(B)$, the set $\{\varphi_n(w) : n \in \mathbb{N}\}$ is an infinite set in $Z(B)$.

Example.

Let $\varphi(z) = 1/(2-z)$. Then φ maps the disk into itself, and $\varphi(1) = \varphi'(1) = 1$, but φ is in the plane translation case. There is NO Blaschke product B so that BH^2 is an invariant subspace for C_φ.

Outline of Proof.

In this case, if w is in $\mathbb{D}$, then $\{\varphi_n(w)\}$ is NOT a Blaschke sequence. ∎

There is space between our results: If J is a singular inner function whose singular measure has no atom, then our work says nothing about possible non-trivial spaces of the form JH^2 that are invariant for C_φ!

THANK YOU!

http://www.math.iupui.edu/˜ccowen/Downloads.html