Composition Operators on Spaces of Analytic Functions, III

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Today, we want to consider more recent developments, some specific unsolved problems, and future directions in the subject.

Circular Symmetry

Wednesday we discussed spectra of composition operators in the various cases. In many of the non-compact cases, there is considerable circular symmetry in the spectra, in fact, all cases except the plane translation case.

- In the non-compact cases with a fixed point inside, the spectra were circularly symmetric except for finitely many points. When the Denjoy-Wolff point is on the boundary, spectra exhibit circular symmetry, except in the plane translation case.

- In one case, (half-plane dilation) the operator is similar to rotates of itself. In the other half-plane case, some parts of spectra have circular symmetry.

- If α_0, α_1, α_2, α_3, $\cdots$ are points of disk satisfying $\varphi(\alpha_k) = \alpha_{k+1}$, then

$$C_\varphi^* \left(\frac{1}{\|K_{\alpha_k}\|} K_{\alpha_k} \right) = \frac{1}{\|K_{\alpha_k}\|} K_{\alpha_{k+1}} = \left(\frac{\|K_{\alpha_{k+1}}\|}{\|K_{\alpha_k}\|} \right) \left(\frac{1}{\|K_{\alpha_{k+1}}\|} K_{\alpha_{k+1}} \right)$$

so $\overline{\operatorname{span}\{K_{\alpha_k}\}}$ is an invariant subspace for C_φ^* and it looks like C_φ^* is a weighted shift on this subspace

In half-plane cases, this can be made into a proof that the restriction of C_φ to an invariant subspace is similar to every rotate of itself.

Conjecture (C. & MacCluer, 1998)

If φ has a fixed point in D and the essential spectral radius of C_φ is positive, there is an invariant subspace for C_φ on which the restriction of C_φ is similar to rotates of itself.

Confirmed in some special cases by R. Wahl, thesis, 1997.

Other Problems

- Hyponormality and subnormality of C_φ or C_φ^*
- Equivalence (unitary equivalence or similarity)
- Commutant of C_φ

Weighted Composition Operators

For φ an analytic map of the disk into itself and ψ a analytic function of the disk into $\mathbb{C}$ the *weighted composition operator* $W_{\psi,\varphi}$ is the operator on $H^2(\mathbb{D})$ given by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$$

for z in $\mathbb{D}$.

Weighted composition operators are an extension of the idea of composition operator, but also have a claim to be more basic because they arose before composition operators in some natural contexts.

The *weighted composition operator* $W_{\psi,\varphi}$ is the operator on H^2 given by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$$

for z in $\mathbb{D}$.

Clearly, if ψ is in $H^\infty(\mathbb{D})$ and φ maps the disk into the disk, then $W_{\psi,\varphi}$ is bounded and if ψ is in H^∞ and C_φ is compact, then $W_{\psi,\varphi}$ is compact also.

Perhaps, surprisingly, these conditions are not necessary!

Because $W_{\psi,\varphi}1 = \psi$, the multiplier ψ must be in H^2,

but it is possible for ψ to be unbounded in H^2 and C_φ non-compact and have $W_{\psi,\varphi}$ bounded or compact.

Considered: conditions for $W_{\psi,\varphi}$ bounded, compact, or self-adjoint,

spectra of compact or self-adjoint weighted composition operators, $\cdots$

Study of weighted composition operators just begun in an organized way $\cdots$

Problems

- Characterizations of boundedness and compactness are not yet in an easily applicable form
- Fredholm weighted composition operators not characterized; same for other classes except self adjoint
- Spectra of compact and self adjoint weighted composition operators completely understood, but no other spectra have been analyzed except special cases.

Adjoints

Descriptions of adjoints of operators are standard parts of the general description of operators.

While the relation $C_\varphi^*(K_\alpha) = K_{\varphi(\alpha)}$ is very useful,

it does not extend easily to a formula for C_φ^*

Theorem (C, 1988).

If $\varphi(z) = \dfrac{az+b}{cz+d}$ *is a non-constant linear fractional map of the unit disk into itself, then*

$$C_\varphi^* = T_g C_\sigma T_h^*$$

where $\sigma(z) = \dfrac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$, $\quad g(z) = \dfrac{1}{-\overline{b}z + \overline{d}}$, *and* $h(z) = cz + d$.

Inner functions can be handled

C_φ^* can be explicitly described as an integral operator,

it is a "folk theorem"

For z in $\mathbb{D}$,

$$(C_\varphi^* f)(z) = \langle C_\varphi^* f, K_z \rangle = \langle f, C_\varphi K_z \rangle = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})} z} \frac{d\theta}{2\pi}$$

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This "formula" is very difficult to actually use $\cdots$ but it was this that allowed me to discover the formula for the linear fractional case.

It worked because linear fractional maps are rational functions and univalent $\cdots$

In the past decade or so, with contributions from several mathematicians, published and not published, we now have a formula for the adjoints of composition operators with symbol a rational function.

Wahl, 1997

Gallardo-Gutiérrez & Montes-Rodríguez, 2003

C. & Gallardo-Gutiérrez, 2005, 2006

Martín & Vukotić, 2006

Hammond, Morehouse, & Robbins, 2008

Bourdon & Shapiro, preprint 2008

We were able to find a formula for C_φ^* for $\varphi(z) = (z + z^2)/2$:

$$(C_\varphi^* f)(z) = \frac{z + \sqrt{z^2 + 8z}}{2\sqrt{z^2 + 8z}} f\left(\frac{z + \sqrt{z^2 + 8z}}{4}\right) - \frac{z - \sqrt{z^2 + 8z}}{2\sqrt{z^2 + 8z}} f\left(\frac{z - \sqrt{z^2 + 8z}}{4}\right)$$

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On the other hand, the formula as a whole DOES make sense for every f in H^2 and defines $C_\varphi^* f$ as a single-valued analytic function!

Definition. (C. & Gallardo-Gutiérrez, 2006)

Let K be a finite subset of $\mathbb{D}$. Suppose σ is an n-valued function that is arbitrarily continuable in $\mathbb{D} \setminus K$ and takes values in $\mathbb{D}$. Suppose ψ is an m-valued analytic function defined and arbitrarily continuable in $\mathbb{D} \setminus K$ with values in $\mathbb{C}$ and bounded.

Plus compatibility condition. *Plus finiteness condition.*

The *multiple valued weighted composition operator* $W_{\psi,\sigma}$ is defined by

$$(W_{\psi,\sigma} f)(z) = \sum_{\textit{all branches}} \psi(z) f(\sigma(z))$$

for z in $\mathbb{D} \setminus K$ and extended to $\mathbb{D}$.

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In the example, $\psi(z) = \dfrac{z \pm \sqrt{z^2 + 8z}}{\pm 2\sqrt{z^2 + 8z}}$ and $\sigma(z) = \dfrac{z \pm \sqrt{z^2 + 8z}}{4}$

These conditions ensure that $(W_{\psi,\sigma}f)(z)$ is bounded on every compact subset of $\mathbb{D}$ and therefore that all the singularities at points of K are removable so that $W_{\psi,\sigma}f$ is analytic in $\mathbb{D}$.

Moreover, if ψ is bounded, that is, there is $M < \infty$ so that

$$\limsup_{|z|\to 1^-} |\psi_j(z)| \leq M$$

for each branch of ψ, then $W_{\psi,\sigma}$ is bounded on H^2 and

$$\|W_{\psi,\sigma}\| \leq M \sqrt{n \sum \frac{1+|\sigma(0)|}{1-|\sigma(0)|}}$$

Theorem. (C. 1978)

Suppose φ *is a finite Blaschke product and let*

$$K = \{\beta \in \mathbb{D} : \varphi(\beta) = \varphi(\alpha) \quad \textit{and} \quad \varphi'(\alpha) = 0\}$$

If S *is a bounded operator on* H^2 *such that* $ST_\varphi = T_\varphi S$*, then* $S = W_{\psi,\eta}$ *where* $\eta = \varphi^{-1} \circ \varphi$ *and* $\psi = G\eta'/\varphi'$ *for some* G *multiple valued, bounded, analytic function on* $\mathbb{D} \setminus K$*.*

Conversely, if G *is a multiple valued, bounded, analytic function on* $\mathbb{D} \setminus K$*,* $\eta = \varphi^{-1} \circ \varphi$*,* $\psi = G\eta'/\varphi'$*, and* $S = W_{\psi,\eta}$*, then* $ST_\varphi = T_\varphi S$*.*

Theorem. (version of Hammond, Moorhouse, Robbins)

Let φ be a rational map of $\mathbb{D}$ into itself. Then for any f in H^2

$$(C_\varphi^* f)(z) = (W_{\psi,\sigma} f)(z) + \frac{f(0)}{1 + \overline{\varphi(\infty)} z}$$

where $W_{\psi,\sigma}$ is multiple valued weighted composition operator induced by

$$\sigma(z) = \left(\overline{\varphi^{-1}(1/\bar{z})}\right)^{-1} \quad \textit{and} \quad \psi(z) = \frac{z\sigma'(z)}{\sigma(z)}$$

$$\textit{and } \varphi(\infty) = \lim_{|w|\to\infty} \varphi(w)$$

Study of multiple valued weighted composition operators has not begun in an organized way $\cdots$

Nothing is known beyond these two examples, nor are other examples related to other parts of functional analysis known.

Composition Operators in Several Variables

Hardy Hilbert space in the ball, $H^2(\mathbf{B}_N)$ where

$$\mathbf{B}_N = \{z \in \mathbb{C}^N : |z| < 1\}$$

Let σ denote normalized surface measure on the sphere, $\{z \in \mathbb{C}^N : |z| = 1\}$.

$$H^2(\mathbf{B}_N) = \{f \text{ analytic in } \mathbf{B}_N : \sup_{0<r<1} \int |f(r\zeta)|^2 \, d\sigma(\zeta) < \infty\}$$

If φ is a map of $\mathbf{B}_N$ into $\mathbf{B}_N$, define C_φ on $H^2(\mathbf{B}_N)$ by

$$(C_\varphi f)(z) = f(\varphi(z))$$

Not all maps φ of $\mathbf{B}_N$ into $\mathbf{B}_N$ give bounded composition operators!

There are even polynomial maps φ for which C_φ is not bounded!

For example, for

$$\varphi(z_1, z_2) = (2z_1z_2, 0)$$

C_φ is unbounded on $H^2(\mathbf{B}_N)$.

While there are Carleson conditions for boundedness and compactness,

these questions are not satisfactorily resolved.

If C_φ is compact on $\mathbf{B}_N$, there is an attracting fixed point a for φ in $\mathbf{B}_N$ and

$$\sigma(C_\varphi) = \{0,1\} \cup \{\lambda : \lambda = \text{ a product of eigenvalues of } \varphi'(a)\}$$

Biggest problem is that we don't know enough functions in several variables.

Theorem. (C. & MacCluer 1994)

Suppose φ is a holomorphic map of $\mathbf{B}_N$ into $\mathbf{B}_N$ that is univalent, $\varphi(0) = 0$, and φ is not unitary on any slice of $\mathbf{B}_N$. If $\Omega < \infty$, then the spectrum of C_φ as an operator on $H^2(\mathbf{B}_N)$ includes the disk

$$\left\{ \lambda : |\lambda| \leq \frac{\tilde{\rho}^N}{\Omega^{N/2}} \right\}$$

where $\tilde{\rho}$ is the essential spectral radius of C_φ on $H^2(\mathbf{B}_N)$ and

$$\Omega = \sup\{\|\varphi'(z)\|^2/|J_\varphi(z)|^2 : z \in B_N\}$$

Theorem.

Suppose φ is a holomorphic map of $\mathbf{B}_N$ into $\mathbf{B}_N$, $\varphi(0) = 0$, and φ is not unitary on any slice of $\mathbf{B}_N$. If C_φ is bounded on $H^2(\mathbf{B}_N)$ then 1 and each number $\overline{\lambda}$ for λ a product of eigenvalues of $\varphi'(0)$ is an eigenvalue of C_φ^*.

Definition.

A map φ will be called a *linear fractional map* if

$$\varphi(z) = (Az + B)(\langle z, C\rangle + D)^{-1}$$

where A is an $N \times N$ matrix, B and C are (column) vectors in $\mathbb{C}^N$, and D is a complex number. We will regard z as a column vector also and $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product on $\mathbb{C}^N$.

Clearly, this is analytic except for z in the hyperplane $\langle z, C\rangle = -D$.

Every automorphism of the ball is a linear fractional map with this definition.

For a linear fractional map $\varphi(z) = (az + b)/(cz + d)$, associate

$$m_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In several variables, for $\varphi(z) = (Az + B)(\langle C, z\rangle + D)^{-1}$, associate

$$m_\varphi = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$

In both cases, $m_\varphi m_\psi = m_{\varphi\circ\psi}$, so we use this as a tool.

If φ is a linear fractional map of $\mathbf{B}_N$ into itself, then C_φ is bounded operator on $H^2(\mathbf{B}_N)$ and compact iff $\overline{\varphi(\mathbf{B}_N)} \subset \mathbf{B}_N$.

The theorem on adjoints of C_φ for φ a linear fractional map of the ball carries over exactly as in one variable.

Problems

- Can every map of the ball to the ball be modeled by a linear fractional map? If not, which can be? Are they special in any way?
- Find spectra, even in special cases.
- Little known about composition operators on other spaces, for example, $H^2(\mathbb{D} \times \mathbb{D})$, or the analogue of the Bergman space.

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Slides posted on webpage:

`www.math.iupui.edu/~ccowen/EPAF09.html`