Composition Operators with Multivalent Symbol*

Rebecca G. Wahl†

University of South Dakota, Vermillion, South Dakota 57069

March 10, 2007

Abstract

If φ is an analytic map of the unit disk D into itself, the composition operator C_φ on the Hardy space $H^2(D)$ is defined by $C_\varphi(f) = f \circ \varphi$. For a certain class of composition operators with multivalent symbol φ, we identify a subspace of $H^2(D)$ on which C_φ^* behaves like a weighted shift. We reproduce the description of the spectrum found in [5] and show for this class of composition operators that the interior of the spectrum is a disk of eigenvalues of C_φ^* of infinite multiplicity.

1. INTRODUCTION

Although composition operators can be defined on any Hilbert space of analytic functions, we consider composition operators on the classical Hardy Hilbert space of analytic functions on the unit disk D, $H^2(D)$. Specifically, $H^2(D)$ is the set of analytic functions f on D for which

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty$$

If φ is an analytic map of the unit disk D into itself, the composition operator C_φ on $H^2(D)$ is defined by $C_\varphi(f) = f \circ \varphi$. Every analytic map of D into D gives a bounded composition operator on $H^2(D)$ and other general properties of these operators are known [2], but the spectral information is incomplete. Kamowitz was the first to study the spectrum of composition operators [5] and found, for C_φ with symbol φ having an interior fixed point

*1991 Mathematics Subject Classification 47B38

†This paper is part of a doctoral thesis written under the direction of Professor Carl C. Cowen, Purdue University

in the disk and analytic in a neighborhood of the closed disk, that the spectrum of C_φ is

$$\{\lambda \ : |\lambda| \leq \rho\} \bigcup \{\varphi'(a)^k : k = 1, 2, \ldots\} \bigcup \{1\}$$

where ρ is a constant that Kamowitz computes in terms of the derivatives of the iterates at the fixed points. Cowen and MacCluer [3], using entirely different proof techniques, reached the same conclusion without assuming analyticity in a neighborhood of the closed disk for composition operators with univalent symbol and identified ρ as the essential spectral radius. However, neither of the Kamowitz or Cowen and MacCluer results classify the points of the spectrum.

We identify in Theorem 6, for a certain class of composition operators with multivalent symbol φ, a subspace of $H^2(D)$ on which C_φ^* behaves like a weighted shift. Using known results for weighted shifts [8]we reproduce the description of the spectrum found in [5] and show for this class of composition operators that the interior of the spectrum is a disk of eigenvalues of C_φ^* of infinite multiplicity.

2. PRELIMINARIES

For $w \in D$, evaluation at w is a bounded linear functional and so, by the Riesz representation theorem, there is a function K_w in $H^2(D)$ that induces this linear functional: $f(w) = \langle f, K_w \rangle$. The function K_w is called the reproducing kernel function. In the Hardy space $H^2(D)$, the reproducing kernel is

$$K_w(z) = \frac{1}{1 - \overline{w}z}$$

and has H^2 norm

$$||K_w(z)|| = \frac{1}{(1 - |w|^2)^{1/2}}$$

The action of the adjoint of a composition operator on reproducing kernel functions is particularly nice. Since

$$\langle f, C_\varphi^* K_\alpha \rangle = \langle C_\varphi f, K_\alpha \rangle = f(\varphi(\alpha)) = \langle f, K_{\varphi(\alpha)} \rangle$$

we have $C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$.

We restrict our attention to composition operators with symbol φ where φ is a multivalent mapping of the unit disk into itself having interior fixed point with derivative zero at this fixed point and exactly one fixed point on $\partial(D)$. The meaning of a boundary fixed point is given in the following definition.

Definition If φ is an analytic mapping of the unit disk into itself and a is a point of the closed unit disk, we say that a is a fixed point of φ if

$$\lim_{r\to 1^-} \varphi(ra) = a$$

Recall that the automorphisms of the unit disk, that is, the one-to-one analytic maps of the disk onto itself, are the functions of the form

$$\varphi(z) = \lambda \frac{a-z}{1-\overline{a}z}$$

where $|\lambda| = 1$ and $|a| < 1$.

Definition We say that a Hilbert space H is automorphism invariant if all composition operators whose symbol is an automorphism are bounded in H.

Since $H^2(D)$ is an automorphism invariant space, we can change variables via an automorphism of the unit disk while preserving the structure of the problem. The automorphism invariance of $H^2(D)$ coupled with the fact that similarity preserves the spectrum of an operator allows us to study the spectrum of the operator C_φ by considering the operator $C_{\psi^{-1}\circ\varphi\circ\psi}$ for any automorphism ψ of the unit disk. We thus assume without loss of generality that the fixed point in the disk is 0 and by rotational similarity that the fixed point on the unit circle is 1. We further limit our focus to multivalent symbols which are $2-\text{to}-1$ near the fixed point 0 and $1-\text{to}-1$ near the fixed point 1 and consider the class of maps

$$\varphi(z) = \frac{(1-2c)z^2}{1-2cz} \text{ for } 0 < c < 1/2 \tag{1}$$

In Section 4 we find a subspace M of $H^2(D)$ on which C_φ^* behaves like a weighted shift for the maps in equation 1. The spectral information for C_φ^* is then determined from spectral results on weighted shifts. The subspace M is given by a coordinate basis of differences of reproducing kernel functions determined by a sequence of backward iterates of $\varphi(z)$.

Definition We say the sequence of points $\{z_j\}_{j=K}^{N}$ in D is an iteration sequence for φ if

$$\varphi(z_j) = z_{j+1}$$

for $K \le j \le N$ where K and N are finite integers or $\pm\infty$. The z_j, $j = K, \ldots, N$, are called iterates of φ. If $\varphi(z_j) = z_{j-1}$, then $\{z_j\}$ is called a backward iteration sequence and, in this case, the z_j are called backward iterates of φ.

Note that for reproducing kernel functions corresponding to a sequence of backward iterates, the adjoint C_φ^* displays a shift property:

$$C_\varphi^*(K_{z_j}) = K_{z_{j-1}}$$

The key to the weighted shift analogy is that the iteration sequence must form an interpolating sequence.

Definition An interpolating sequence is a sequence $\{z_j\}$ in the disk such that for any bounded sequence of complex numbers $\{c_j\}$, there is a bounded analytic function f on D with $f(z_j) = c_j$.

If a set of points forms an interpolating sequence then a theorem on interpolation in H^2 due to Shapiro and Shields [7, Theorem 2], and a form of the theorem due to Cowen [1, Corollary 3.3], can be applied to show that the corresponding normalized reproducing kernels form an almost orthonormal set. That is, there exists a bounded operator with bounded inverse taking an orthonormal set onto this sequence of normalized reproducing kernels. In the present case, we show that C_φ^* on a coordinate basis of differences of such vectors is similar to a weighted shift acting on an orthonormal basis.

We require the following notions from geometric function theory.

Definition For ζ on the unit circle and $\alpha > 1$ we define a nontangential approach region at ζ by

$$\Gamma(\zeta, \alpha) = \{z \in D : |z - \zeta| < \alpha(1 - |z|)\}$$

Definition A function f is said to have a nontangential limit at ζ if $\lim_{z\to\zeta}$ exists for each nontangential region $\Gamma(\zeta, \alpha)$.

Definition We say φ has a finite angular derivative at ζ on the unit circle if there is a point η on the unit circle so that $\dfrac{\varphi(z) - \eta}{z - \zeta}$ has a finite nontangential limit as $z \to \zeta$. When it exists (as a finite complex number), this limit is denoted $\varphi'(\zeta)$.

Theorem 1 *(Julia-Carathéodory Theorem) For $\varphi : D \to D$ analytic and ζ in ∂D, the following are equivalent:*

(1) $d(\zeta) = \liminf_{z\to\zeta}(1 - |\varphi(z)|)/(1 - |z|) < \infty$ *, where the limit is taken as z approaches ζ unrestrictedly in D.*

(2) *φ has finite angular derivative $\varphi'(\zeta)$ at ζ.*

(3) *Both φ and φ' have (finite) nontangential limits at ζ, with $|\eta| = 1$ for $\eta = \lim_{r\to 1} \varphi(r\zeta)$.*

Moreover, when these conditions hold, we have $lim_{r\to 1}\varphi'(r\zeta) = \varphi'(\zeta) = d(\zeta)\overline{\zeta}\eta$ *and* $d(\zeta)$ *is the nontangential limit* $\lim_{z\to\zeta}(1-|\varphi(z)|)/(1-|z|)$.

3. GEOMETRIC PROPERTIES OF $\varphi(z) = \frac{(1-2c)z^2}{1-2cz}$ FOR $0 < c < 1/2$

We now present several lemmas describing the behavior of the mapping $\varphi(z)$. These lemmas culminate with Lemma 4 which asserts the existence of a sequence of backward iterates for φ that interpolate. In Lemma 2 we show that there exists a branch of the square root function so that φ^{-1} is defined and univalent on the disk $D_1 = \{z \in \mathrm{C} : |z-1| < 1\}$ and maps D_1 into itself. We then apply Koenig's model for iteration [6] to φ^{-1} on D_1 in the proof of Lemma 4.

Lemma 2 *For* $\varphi(z) = \frac{(1-2c)z^2}{1-2cz}$, $0 < c < 1/2$, *the mapping* $\varphi(z)$ *is univalent on* $D_1 = \{z \in \mathrm{C} : |z-1| < 1\}$. *Moreover, a branch of* $\varphi^{-1}(z)$ *can be defined that is univalent on* D_1 *and maps* D_1 *into itself.*

Proof. Note that $\varphi(z) = \varphi(w)$

$$\begin{aligned}
&\Leftrightarrow & \tfrac{(1-2c)z^2}{1-2cz} = \tfrac{(1-2c)w^2}{1-2cw} \\
&\Leftrightarrow & 2c(zw^2 - wz^2) + z^2 - w^2 = 0 \\
&\Leftrightarrow & 2cwz(w-z) + (z+w)(z-w) = 0 \\
&\Leftrightarrow & [w(1-2cz) + z](w-z) = 0
\end{aligned}$$

so that $\varphi(z) = \varphi(w)$ if and only if $z = w$ or $w = \dfrac{z}{2cz-1}$. Thus, to show that φ is univalent on D_1 it is enough to show that $w = \dfrac{z}{2cz-1}$, $0 < c < 1/2$, is not in D_1 for $z \in D_1$. Now

$$\begin{aligned}
\left|\frac{z}{2cz-1} - 1\right| &= \left|\frac{1+(1-2c)z}{2cz-1}\right| \\
&= \left|\frac{1+(1-2c)(z-1)+(1-2c)}{-1+2c(z-1)+2c}\right| \\
&= \left|\frac{2-2c+(1-2c)(z-1)}{2c-1+2c(z-1)}\right| \\
&= \left|\frac{1-2c}{2c}\right|\left|\frac{\frac{2-2c}{1-2c}+(z-1)}{\frac{-(1-2c)}{2c}+(z-1)}\right| \\
&= \left|\frac{1-2c}{2c}\right|\left|\frac{\frac{2-2c}{1-2c}+(z-1)}{\frac{1-2c}{2c}-(z-1)}\right|
\end{aligned}$$

and since $1 < \frac{1}{1-2c} < \frac{2-2c}{1-2c}$ and since $|z-1| < 1$, we have that the modulus of the numerator of the second factor is least when $z = 0$ so that

$$\left|\frac{2-2c}{1-2c} + (z-1)\right| > \left|\frac{2-2c+2c-1}{1-2c}\right| = \left|\frac{1}{1-2c}\right|$$

By the triangle inequality, we have that the denominator of the second factor has modulus less than $\frac{1}{2c}$ since

$$\left|\frac{1-2c}{2c} - (z-1)\right| \leq \left|\frac{1-2c}{2c}\right| + |z-1| < \frac{1-2c}{2c} + 1 = \frac{1}{2c}$$

so that

$$\left|\frac{z}{2cz-1} - 1\right| > 1$$

Hence, φ is univalent on D_1.

Since φ is function of degree two on the complex plane, if $|w-1| < 1$ then there exist complex numbers z_1, z_2 such that $\varphi(z_1) = \varphi(z_2) = w$. Note if $\varphi(z) = \frac{(1-2c)z^2}{1-2cz} = w$, then $(1-2c)z^2 - (1-2c)w = 0$. This can be written in the form

$$(z-1)^2 + \left[\frac{2(1-2c) + 2c(w-1)}{1-2c}\right](z-1) - (w-1) = 0$$

Now, this quadratic equation has two roots whose product is the constant term. Hence, for every w there exist z_1 and z_2 such that $\varphi(z_1) = \varphi(z_2) = w$ and if $|w-1| < 1$ then $|z_1-1||z_2-1| = |w-1| < 1$. Since $|z_1-1||z_2-1| < 1$ at least one of the points z_1 or z_2 is in the disk D_1. Our task is to show that only one of these points is in D_1. Suppose, $|z_1-1| < 1$. As was shown above in the proof of the univalence of φ, $z_2 = \dfrac{z_1}{2cz_1-1}$ and we have by that calculation that $|z_2-1| > 1$. Thus, we can define a mapping $p(w) = z_1$. Note that p maps D_1 into D_1 with $p(1) = 1$, $\varphi(p(w)) = w$ for all w in D_1 and thus, $p'(1) = \dfrac{1}{\varphi'(1)}$. We shall denote $p(z)$ by $\varphi^{-1}(z)$. ■

Let $Arg(z)$ denote the value of the argument of $z \neq 0$ with $0 \leq Arg(z) < 2\pi$ and let Re(z) and Im(z) denote the real and imaginary parts of the complex number z. In the next lemma we show that $Arg(\varphi(z)) > 2Arg(z)$ for points z in the disk with $0 < Arg(z) < \pi/2$, and that φ maps points z in the disk with $\pi/2 < Arg(z) < \pi$ into the lower half-disk $\{z \in D : \text{Im(z)} < 0\}$. These properties of φ are used in the proof of Lemma 4 to identify the sequence of backward iterates of φ in D that form the interpolation sequence used in the weighted shift analogy.

Lemma 3 *Let* $\varphi(z) = \dfrac{z^2(1-2c)}{1-2cz}$ *for* $0 < c < \frac{1}{2}$. *For the set of points* $\{z \in D : 0 < Arg(z) < \frac{\pi}{2}\}$ *we have that* $Arg(\varphi(z)) > 2Arg(z)$ *and for the points* $\{z \in D : \frac{\pi}{2} < Arg(z) < \pi\}$ *we have* $\pi < Arg(\varphi(z)) < 2\pi$.

Proof. Consider $z \in D$ with $0 < Arg(z) < \frac{\pi}{2}$. Since $\dfrac{1}{2c} > 1$,

$$\mathrm{Re}(\frac{1}{2\mathrm{c}} - \mathrm{z}) = \frac{1}{2\mathrm{c}} - \mathrm{Re}(\mathrm{z}) > 0$$

and since $\mathrm{Im}(\frac{1}{2\mathrm{c}} - \mathrm{z}) = \mathrm{Im}(-\mathrm{z}) < 0$ we see that $\frac{1}{2c} - z$ is in quadrant IV of the complex plane. Hence, for $z \in D$ with $0 < Arg(z) < \frac{\pi}{2}$, we have

$$\frac{3\pi}{2} < Arg(\frac{1}{2c} - z) < 2\pi$$

or

$$-2\pi < -Arg(\frac{1}{2c} - z) < -\frac{3\pi}{2}$$

Noting that $0 < 2Arg(z) < \pi$ we have

$$-2\pi < 2Arg(z) - Arg(\frac{1}{2c} - z) < -\frac{\pi}{2}$$

and

$$0 < 2Arg(z) - Arg(\frac{1}{2c} - z) + 2\pi < 2\pi$$

For any branch of the argument function, and so for $Arg(z)$, we have $w = |w|e^{iArg(w)}$ and since multiplication by a scalar does not change the argument of a point we have

$$\begin{aligned}
\varphi(z) &= \frac{z^2(1-2c)}{1-2cz} \\
&= |\varphi(z)|e^{iArg\left(\frac{z^2(1-2c)}{1-2cz}\right)} \\
&= |\varphi(z)|\frac{e^{iArg(z^2(1-2c))}}{e^{iArg(1-2cz)}} \\
&= |\varphi(z)|\frac{e^{iArg(z^2)}}{e^{iArg(\frac{1}{2c}-z)}} \\
&= |\varphi(z)|e^{i[2Arg(z)-Arg(\frac{1}{2c}-z)]}
\end{aligned}$$

By the periodicity of the exponential function

$$\varphi(z) = |\varphi(z)|e^{i[2Arg(z) - Arg(\frac{1}{2c} - z) + 2\pi]}$$

Recalling that $0 < 2Arg(z) - Arg(\frac{1}{2c} - z) + 2\pi < 2\pi$ we have

$$Arg(\varphi(z)) = 2Arg(z) - Arg(\frac{1}{2c} - z) + 2\pi$$

and since $-2\pi < -Arg(\frac{1}{2c} - z) < -3\pi/2$ we have $0 < 2\pi - Arg(\frac{1}{2c} - z)$ or

$$Arg(\varphi(z)) = 2Arg(z) + (2\pi - Arg(\frac{1}{2c} - z)) > 2Arg(z)$$

for $0 < Arg(z) < \frac{\pi}{2}$ as desired.

To prove that φ maps points z in the disk with $\pi/2 < Arg(z) < \pi$ into the lower half-disk $\{z \in D : \mathrm{Im(z)} < 0\}$, we need only consider

$$\mathrm{Im}(\varphi(\mathrm{z})) = \frac{(1-2\mathrm{c})}{2\mathrm{c}} \frac{\frac{\mathrm{xy}}{\mathrm{c}} - \mathrm{yx}^2 - \mathrm{y}^3}{(\frac{1}{2\mathrm{c}} - \mathrm{x})^2 + \mathrm{y}^2}$$

where $z = x + iy$ with $y > 0$ and $x < 0$. Note that

$$\begin{aligned} \mathrm{Im}(\varphi(\mathrm{z})) < 0 \quad &\Leftrightarrow \quad \frac{xy}{c} - yx^2 - y^3 < 0 \\ &\Leftrightarrow \quad y(\frac{x}{c} - x^2 - y^2) < 0 \\ &\Leftrightarrow \quad \frac{x}{c} - x^2 - y^2 < 0 \end{aligned}$$

Since $x < 0$ we have $\frac{x}{c} - x^2 - y^2 < 0$ and so φ maps points in the disk with $\pi/2 < Arg(z) < \pi$ into the lower half-disk. ■

Lemma 4 *Let* $\varphi(z) = \dfrac{z^2(1-2c)}{1-2cz}$ *for* $0 < c < \frac{1}{2}$. *There exists* z_j, $j = 0, 1, \ldots$ *such that*

(i) $-1 < z_0 < 0$

(ii) $Im(z_j) > 0$ *for* $j = 1, 2, \ldots$

(iii) $\varphi(z_j) = z_{j-1}$ *for* $j = 1, 2, \ldots$

(iv) $\lim_{j\to\infty} z_j = 1$

(v) $\{z_j\}$ *form an interpolating sequence*

Proof. Since $\varphi(z)$ is univalent on the disk $D_1 = \{z : |z-1| < 1\}$ and $\varphi^{-1}(z)$ maps D_1 into itself, Koenigs' model for iteration applied to $\varphi^{-1}(z)$ on D_1 assures that there exists an analytic mapping $\sigma(z)$ which is univalent on D_1 and maps D_1 into the plane $\Omega = \mathrm{C}$ with $\sigma(1) = 0$ and $\sigma'(1) = 1$ and for all $z \in D_1$ with $\varphi(z) \in D_1$ we have

$$\sigma(\varphi^{-1}(z)) = (\varphi^{-1})'(1)\sigma(z) = \frac{1-2c}{2-2c}\sigma(z)$$

Thus, for $w = \varphi^{-1}(z)$

$$\sigma(w) = \frac{1-2c}{2-2c}\sigma(\varphi(w))$$

or

$$\sigma(\varphi(w)) = \varphi'(1)\sigma(w) = \frac{2-2c}{1-2c}\sigma(w)$$

for $\varphi(w) \in D_1$. Note that this relation gives a relation for the iterates of φ whenever $\varphi_n(z) \in D_1$:

$$\sigma(\varphi_n(z)) = \left(\frac{2-2c}{1-2c}\right)^n \sigma(z)$$

Let $a = \frac{2-2c}{1-2c}$. Since $\sigma(z)$ is continuous on D_1, $\sigma(z)$ maps the half-neighborhood $N_\epsilon = \{z \in D \cap D_1 : |z-1| < \epsilon\}$ onto a half-neighborhood of 0 in the plane. Since φ^{-1} is univalent on $D \cap D_1$, the theory of the model implies that σ is univalent on $D \cap D_1$ and since φ maps the interval $(0,1)$ into itself the normalization $\sigma'(1) = 1$ implies that $\sigma(\mathrm{R}) \subset \mathrm{R}$ and σ has a Taylor series near 1 of the form

$$\sigma(z) = (z-1) + b_2(z-1)^2 + b_3(z-1)^3 + \ldots$$

Thus for $\epsilon > 0$ small enough, we can choose u in the half-neighborhood of 0 in the plane $\sigma(N_\epsilon)$ with $au \in \sigma(N_\epsilon)$, $\mathrm{Re(u)} < 0$ and $\mathrm{Im(u)} > 0$. Parametrize the line segment from 0 to au, $[0, au]$, by tu for $0 \le t \le a$.

Suppose $u = \sigma(p_0)$ for some $p_0 \in D \cap D_1$. Then the intertwining relation $\sigma(\varphi(z)) = a\sigma(z)$ implies $au = a\sigma(p_0) = \sigma(\varphi(p_0)) = \sigma(p_1)$ where $p_1 = \varphi(p_0)$. Thus, the image of the line segment $[0, au]$ under the mapping $\sigma^{-1}(z)$ is a curve in the disk from 1 to p_1 through p_0 with $p_0, p_1 \in N_\epsilon$ that can be naturally parametrized by

$$\gamma(t) = \sigma^{-1}(tu)$$

for $0 \le t \le a$.

Note that we have

$$\gamma(ar) = \varphi(\gamma(r)) \text{ for } 0 \leq r \leq 1$$

Indeed, if $0 \leq r \leq 1$ then $0 \leq aru \leq au$ and letting $\sigma^{-1}(ru) = z$, so that $ru = \sigma(z)$, we have by Koenigs' model $\varphi(\sigma^{-1}(ru)) = \sigma^{-1}(aru)$. Thus,

$$\begin{aligned} \gamma(ar) &= \sigma^{-1}(aru) \\ &= \varphi(\sigma^{-1}(ru)) \\ &= \varphi(\gamma(r)) \end{aligned}$$

For $s > 1$, define γ inductively by

$$\gamma(a^s) = \varphi(\gamma(a^{s-1})) = p_s$$

The analyticity of σ^{-1} guarantees smoothness of $\gamma(as)$ for $0 \leq s \leq 1$ and the smoothness of $\gamma(as)$ for $0 \leq s \leq 1$ and the analyticity of φ insure that $\gamma : (0, \infty) \to D$ is a continuous curve in D mapped onto itself by φ.

Now, by Lemma 3, $Arg\varphi(z) > 2Arg(z)$ for all $z \in D$ with $0 < Arg(z) < \pi/2$, and points $z \in D$ with $\pi/2 < Arg(z) < \pi$ are mapped into the lower half-disk, so there exists a least $r_0 > 1$ such that $\gamma(a^{r_0}) \in (-1, 0)$. By the definition of γ, $\gamma(a^{r_0}) = \varphi(\gamma(a^{r_0-1})) = p_{r_0}$. Let $z_0 = \gamma(a^{r_0})$ and define for $j \geq 0$, $z_j = \gamma(a^{r_0-j})$.

By the model for iteration applied to $\varphi^{-1}(z)$ on the disk $D_1 = \{z : |z-1| < 1\}$, we have

$$\lim_{j\to\infty} z_j = 1$$

nontangentially. More explicitly, since $\gamma(a^{r_0-j})$ is on $\sigma^{-1}([0, au])$ for $j \geq r_0$ and since σ^{-1} maps the negative real axis into the interval $(0, 1)$ conformally at 0, it maps $[0, au]$ onto a curve that makes an acute angle with $(0, 1)$. Thus, the backward iterates of approach 1 nontangentially. This nontangential convergence and a form, due to Cowen [1, Corollary 3.3], of a theorem on interpolation in H^2 due to Shapiro and Shields [7, Theorem 2], assure that the points $\{z_j\}_{j=0}^{\infty}$ form an interpolating sequence. ■

The reason for identifying z_0, $-1 < z_0 < 0$, as the initial point of the sequence of backward iterates for φ instead of taking the full sequence of iterates is, since φ maps the negative real axis into the positive real axis, the forward iterates of z_0 under φ converge to zero and so do not form an interpolating sequence.

By symmetry of the symbol φ, the complex conjugates $\{\overline{z}_j\}_{j=0}^{\infty}$ of the iteration sequence identified in Lemma 4 also converge nontangentially to 1 in D and $-1 < \varphi_{r_0}(\overline{p_0}) < 0$ where $\overline{p}_0$ is the complex conjugate of the point p_0 identified in the proof of Lemma 4. Moreover, the points $\{\overline{z}_j\}_{j=0}^{\infty}$ form an interpolating sequence. Since $\{z_j\}$ and $\{\overline{z}_j\}$ are interpolating sequences contained in two separate angular sectors, a fact on interpolating sequences applies [4] and asserts that the union $\{z_j, \overline{z}_j\}$ forms an interpolating sequence.

Let

$$k_j = \frac{K_{z_j}}{||K_{z_j}||} = \frac{\sqrt{1-|z_j|^2}}{1-\overline{z}_j z}$$

denote the normalized reproducing kernel functions corresponding to the iteration sequence $\{z_j\}$ and denote by $\tilde{k}_j$ the corresponding normalized reproducing kernel functions $\frac{K_{\overline{z}_j}}{||K_{\overline{z}_j}||}$. Recall that if a set of points forms an interpolating sequence then the corresponding normalized reproducing kernels form an almost orthonormal set [1]. Hence, the union $\{k_j, \tilde{k}_j\}$ forms an almost orthonormal set.

The following calculation shows that the set of differences

$$\{k_1 - \tilde{k}_1, k_2 - \tilde{k}_2, \ldots\}$$

forms an almost orthonormal set. Recall that a set of vectors is said to be an almost orthonormal set if there exists a bounded operator with bounded inverse mapping the set onto an orthonormal set.

Lemma 5 *Suppose the vectors $\{e_j\}$, $j = 0, 1, 2, \ldots$, form an almost orthonormal set. Then the vectors $\{e_{2j} - e_{2j+1}\}$, $j = 0, 1, 2, \ldots$, form an almost orthonormal set.*

Proof. Let S denote the bounded operator with bounded inverse which maps the set $\{e_j\}$ onto an orthonormal set, say $S(e_j) = f_j$. Then $T = \frac{1}{\sqrt{2}}S$ is a bounded map of $\{e_{2j} - e_{2j+1}\}$ onto the orthonormal set $\{(1/\sqrt{2})(f_{2j} - f_{2j+1})\}$, $j = 0, 1, 2, \ldots$, with bounded inverse $T^{-1} = \sqrt{2}S^{-1}$. Indeed,

$$\begin{aligned} &< T(e_{2j} - e_{2j+1}), T(e_{2j} - e_{2j+1}) > \\ &= \frac{1}{\sqrt{2}} < f_{2j} - f_{2j+1}, f_{2j} - f_{2j+1} > \\ &= \frac{1}{\sqrt{2}} < \frac{1}{\sqrt{2}} f_{2j}, \frac{1}{\sqrt{2}} f_{2j} > + \frac{1}{\sqrt{2}} < \frac{1}{\sqrt{2}} f_{2j+1}, \frac{1}{\sqrt{2}} f_{2j+1} > \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}+\frac{1}{2} \\
&= 1
\end{aligned}
$$

and, unless $k = j$, we have

$$< T(e_{2j} - e_{2j+1}), T(e_{2k} - e_{2k+1}) >= 0$$

■

4. THE WEIGHTED SHIFT ANALOGY FOR C_φ^* AND SPECTRAL INFORMATION

By Lemma 5, we know that the vectors

$$\{k_j - \tilde{k}_j\}_{j=1}^{\infty} = \left\{ \frac{K_{z_j}}{||K_{z_j}||} - \frac{K_{\overline{z}_j}}{||K_{\overline{z}_j}||} \right\}_{j=1}^{\infty}$$

form an almost orthonormal set. We now show that C_φ^* on a coordinate basis of these vectors is similar to a weighted shift acting on an orthonormal basis. Specifically, in the following theorem we identify a subspace M of $H^2(D)$ for which the restriction of C_φ^* to M is similar to a weighted shift. Since operators which are similar have identical spectrum, the spectrum of the operator C_φ^*, $\varphi(z) = \dfrac{z^2(1-2c)}{1-2cz}$, $0 < c < \frac{1}{2}$ is obtained through known results on weighted shifts [8].

Theorem 6 *Let* $\varphi(z) = \dfrac{z^2(1-2c)}{1-2cz}$, $0 < c < \frac{1}{2}$ *and let* C_φ *be the associated composition operator on* $H^2(D)$. *Let* $k_j = \dfrac{K_{z_j}}{||K_{z_j}||}$ *and* $\tilde{k}_j = \dfrac{K_{\overline{z}_j}}{||K_{\overline{z}_j}||}$ *denote the normalized reproducing kernel functions corresponding to the iteration sequences* $\{z_j\}$ *and* $\{\overline{z}_j\}$ *identified before Lemma 5. Then the subspace*

$$M = \overline{sp}\left\{ \frac{K_{z_j}}{||K_{z_j}||} - \frac{K_{\overline{z}_j}}{||K_{\overline{z}_j}||} \right\}_{j=1}^{\infty}$$

is such that $C_{\varphi_{|M}}^*$ *is similar to the backward weighted shift with weight sequence*

$$\left\{ \left(\frac{1-|z_j|^2}{1-|z_{j-1}|^2} \right)^{1/2} \right\}_{j=1}^{\infty}$$

Proof. Note the action of the adjoint C_φ^* on the differences of normalized reproducing kernels:

$$\begin{aligned}
C_\varphi^*(k_j - \tilde{k}_j) &= \frac{1}{||K_{z_j}||} C_\varphi^*(K_{z_j} - K_{\overline{z}_j}) \\
&= \frac{1}{||K_{z_j}||}(K_{\varphi(z_j)} - K_{\varphi(\overline{z}_j)}) \\
&= \frac{1}{||K_{z_j}||}(K_{z_{j-1}} - K_{\overline{z}_{j-1}}) \\
&= \frac{||K_{z_{j-1}}||}{||K_{z_j}||} \frac{(K_{z_{j-1}} - K_{\overline{z}_{j-1}})}{||K_{z_{j-1}}||} \\
&= \frac{\sqrt{1-|z_j|^2}}{\sqrt{1-|z_{j-1}|^2}}(k_{j-1} - \tilde{k}_{j-1})
\end{aligned}$$

Thus, C_φ^* shifts the basis for M, M is an invariant subspace for C_φ^* and since the basis for M forms an almost orthonormal set the restriction of C_φ^* to M is similar to a backward weighted shift with weight sequence $\left\{ \left(\frac{1-|z_j|^2}{1-|z_{j-1}|^2} \right)^{1/2} \right\}_{j=1}^{\infty}$.

■

Note that the subspace $M \neq H^2(D)$. Indeed, the constant functions are orthogonal in $H^2(D)$ to the vectors $k_j - \tilde{k}_j$ for all nonnegative integers j.

The next two corollaries follow directly from Theorem 6 and known results on weighted shifts [8]. Corollary 7 answers a conjecture of C.Cowen's posed in 1994 at the Joint Mathematics Meetings in Richmond, Virginia.

Corollary 7 *Let* $\varphi(z) = \dfrac{z^2(1-2c)}{1-2cz}$,$0 < c < \frac{1}{2}$, *and let* C_φ *be the associated composition operator on* $H^2(D)$. *Then we have that*

$$C^*_{\varphi_{|M}} \sim e^{i\theta} C^*_{\varphi_{|M}}$$

where M *is the subspace identified in Theorem 6.*

Proof. This is the content of Corollary 2 to Proposition 1 of [8]. ■

Corollary 8 *Let* $\varphi(z) = \dfrac{z^2(1-2c)}{1-2cz}$,$0 < c < \frac{1}{2}$, *and let* C_φ *be the associated composition operator on* $H^2(D)$. *Let* $a = \varphi'(1) = \frac{2-2c}{1-2c}$. *Then*

$$|\lambda| < \frac{1}{\sqrt{a}} \Rightarrow \lambda \text{ is an eigenvalue of } C_\varphi^*$$

Proof. Recall that we have $z_k = \varphi(z_{k+1})$ for the interpolating sequence identified in Lemma 4. Since $\lim_{k\to\infty} z_k = 1$ nontangentially, by the Julia-Carathéodory Theorem we have

$$\lim_{k\to\infty} \frac{1-|z_k|^2}{1-|z_{k+1}|^2} = \lim_{k\to\infty} \frac{1-|\varphi(z_{k+1})|^2}{1-|z_{k+1}|^2} = \varphi'(1) = a$$

Theorem 4 of [8] then completes the proof. ■

References

[1] C.C. Cowen. Composition operators on H^2. *J.Operator Theory*, 9:77–106, 1983.

[2] C.C. Cowen and B.D. MacCluer. *Composition operators on spaces of analytic functions*. CRC Press, New York, 1994.

[3] C.C. Cowen and B.D. MacCluer. Spectra of some composition operators. *J. Functional Analysis*, 125:223–251, 1994.

[4] J.B. Garnett. *Bounded analytic functions*. Academic Press, New York, 1981.

[5] H. Kamowitz. The spectra of composition operators on H^p. *J. Functional Analysis*, 18:132–150, 1975.

[6] G. Koenigs. Recherches sur les intégrals de certaines équations fonctionnelles. *Ann. Sci. École Norm. Sup. (Sér. 3)*, 1:supplément,3–41, 1884.

[7] H.S. Shapiro and A.L. Shields. On some interpolation problems for analytic functions. *American J. Math*, 83:513–532, 1961.

[8] A.L. Shields. Weighted shift operators and analytic function theory. In *Topics in Operator Theory*, pages 49–128. Math. Surveys, vol. 13, Amer. Math. Soc., Providence, 1974.