SPECTRAL THEORY OF COMPOSITION OPERATORS ON HARDY SPACES OF THE UNIT DISC AND OF THE UPPER HALF-PLANE

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SPECTRAL THEORY OF COMPOSITION OPERATORS ON HARDY SPACES OF THE UNIT DISC AND OF THE UPPER HALF-PLANE

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ABSTRACT

SPECTRAL THEORY OF COMPOSITION OPERATORS ON HARDY SPACES OF THE UNIT DISC AND OF THE UPPER HALF-PLANE

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In this thesis we study the essential spectrum of composition operators on the Hardy space of the unit disc and of the upper half-plane. Our starting point is the spectral analysis of the composition operators induced by translations of the upper half-plane. We completely characterize the essential spectrum of composition operators that are induced by perturbations of translations.

Keywords: Essential Spectrum, Composition Operators, Hardy Spaces.

ÖZ

BÎRÎM DÎSKÎN VE ÜST YARI DÜZLEMÎN HARDY UZAYLARI ÜZERÎNDEKÎ BÎLEŞKE OPERATÖRLERÎNÎN SPEKTRAL TEORÎSÎ

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Bu tezde birim diskin ve üst yarı düzlemin Hardy uzayları üzerindeki bileşke operatörlerinin esas spektrumunu çalışıyoruz. Başlangıç örneğimiz üst yarı düzlemde ötelemeler tarafından türetilen bileşke operatörlerinin spektral teorisidir. Ötelemelerin pertürbasyonları tarafından türetilen bileşke operatörlerinin esas spektrumlarını tamamiyle karakterize ediyoruz.

Anahtar Kelimeler: Esas Spektrum, Bileşke Operatörleri, Hardy uzayları.

To my parents

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CHAPTER 1

INTRODUCTION

In this thesis we focus on the essential spectrum of composition operators on the Hardy spaces of the unit disc and the upper half-plane. On the unit disc we concentrate on the case where the inducing function φ has Denjoy-Wolff point a on the boundary with derivative $\varphi'(a) = 1$ there. We give complete characterization of the essential spectra of a class of composition operators that is included in the subcase "plane/translation" as Cowen and McCluer call it in [7, pp.300]. In [7, pp.300] these authors write about this case as follows: "... is that the spectra for C_φ when φ is in the plane/translation case need not show any circular symmetry. This case is poorly understood; we present a class of examples that permit calculation but do not suggest plausible general techniques". As these experts say the characterization of the spectrum and of the essential spectrum of such composition operators is not yet completed. For more information we refer the reader to [7,pp.299-304].

We obtain the complete characterization of the essential spectrum for C_φ's on $H^2(\mathbb{H})$, the Hardy space of the upper half-plane and on $H^\infty(\mathbb{H})$,the space of bounded analytic functions of the upper half-plane for which the inducing function φ satisfies the conditions that φ is analytic across the boundary, $\psi(z) = \varphi(z) - z$ is a bounded analytic function on $\mathbb{H}$ and the closure of the image of $\mathbb{H}$ under ψ is compact in $\mathbb{H}$. Such maps can be considered as perturbations of the translations $t(z) = z + \alpha$, where $\Im(\alpha) > 0$. We find out the essential spectrum of the composition operator on $H^p(\mathbb{H})$ induced by such a holomorphic map is given by

$$\sigma_e(C_\varphi) = \{e^{i\psi(x)t} : x \in \mathbb{R} \qquad t \in [0, \infty)\} \cup \{0\}.$$

This set consists of a collection of spiral curves that start from 1 and clusters at 0. Furthermore if $\lim_{x\to\infty} \psi(x) = b_0 \in \mathbb{H}$ exists then the essential spectrum of the composition operator on $H^\infty(\mathbb{H})$ induced by φ is given by

$$\sigma_e(C_\varphi) = \{e^{ib_0 t} : t \in [0, \infty)\} \cup \{0\}.$$

This is a spiral curve that starts from 1 and clusters at 0. An example of such a map which is not a translation can be easily found as $\varphi(z) = z+r((z-i)/(z+i))+\alpha$ where $r > 0$ and $\Im(\alpha) > r$. We also characterize the essential spectrum of the composition operators C_φ on $H^2(\mathbb{D})$, the Hardy space of the unit disc and on $H^\infty(\mathbb{H})$,the space of bounded analytic functions of the unit disc for which the conjugate $\tau^{-1}\varphi\tau$ of φ with respect to the Cayley transform $\tau(z) = (z-i)/(z+i)$ satisfies the above conditions. It is not difficult to see that such a self-map φ of the unit disc has the Denjoy-Wolff fixed point at 1, and the derivative $\varphi'(1) = 1$.

Now we introduce the notation that we will use throughout. If S is a compact Hausdorff topological space, $C(S)$ will denote the space of all continuous functions on S. If X is a Banach space, $K(X)$ will denote the space of all compact operators on X, and $\mathcal{B}(X)$ will denote the space of all bounded linear operators on X. The open unit disc will be denoted by $\mathbb{D}$, the open upper half-plane will be denoted by $\mathbb{H}$, the real line will be denoted by $\mathbb{R}$ and the complex plane will be denoted by $\mathbb{C}$. For any $z \in \mathbb{C}$, $\Re(z)$ will denote the real part, and $\Im(z)$ will denote the imaginary part of z respectively. By $\mathcal{S}(\mathbb{R})$ we will denote the Schwartz space of indefinitely differentiable functions f on $\mathbb{R}$ such that for each n,$m \geq 1$, the function $t^n f^{(m)}(t)$ is bounded on $\mathbb{R}$. Note that $\mathcal{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for all $1 \leq p < \infty$.

The essential spectrum of an operator T acting on a Banach space X is the spectrum of the coset of T in the Calkin algebra $\mathcal{B}(X)/K(X)$, the algebra of bounded linear operators modulo compact operators. We will use the notation $\sigma(T)$ for the spectrum of T and $\sigma_e(T)$ for the essential spectrum of T.

A bounded linear operator T on a Banach space X is said to be Fredholm if the kernel $ker(T)$ of T is a finite dimensional subspace of X and the cokernel $X/ran(T)$ is finite dimensional. The Fredholm index $i(T)$ of T , is the dimension of the kernel of T minus the codimension of the range of T. The well known Atkinson's theorem identifies the essential spectrum of T as the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is not a Fredholm operator.

The thesis is organized as follows:
In chapter 2 we give the basic definitions. We establish an isometric isomorphism between $H^p(\mathbb{D})$ and $H^p(\mathbb{H})$. Upon conjugation under this isometric isomorphism, C_φ on $H^p(\mathbb{D})$ becomes a weighted composition operator on $H^p(\mathbb{H})$.

In chapter 3 we use the Cauchy kernel to represent the composition operator acting on $H^p(\mathbb{H})$ as an integral operator acting on $L^p(\mathbb{R})$. Here we consider the functions in $H^p(\mathbb{H})$ inside $L^p(\mathbb{R})$ via their boundary values.

In chapter 4 we treat the Paley-Wiener theorem, which characterizes the $L^p(\mathbb{R})$ functions that are boundary values of functions in $H^p(\mathbb{H})$.

In chapter 5 we analyze the spectra of composition operators on $H^2(\mathbb{H})$ induced by translations of the upper half-plane. While the results are well known ([9],[13]) our development proceeds along different lines from those appearing in the literature. We use the Cauchy kernel to represent our composition operator as an operator of convolution type on $L^p(\mathbb{R})$. Then for $p = 2$, using the Fourier transform and Paley-Wiener theorem we convert it to a multiplication operator on $L^2([0, \infty))$.

In chapter 6 we treat algebras with symbols. We remind the definition of algebra with symbol given in [6]. We modify the definition in [6] since it is designed for studying operators in $\mathcal{B}(L^2)$. Whereas we study operators in $\mathcal{B}(H^2)$.

In chapter 7 we construct an algebra $\mathcal{A}_p$ of operators generated by multiplication operators, convolution operators and compact operators where $1 < p < \infty$. We observe that the commutators of these two types of operators are compact. So our algebra $\mathcal{A}_p/K(H^p)$ of operators is a commutative subalgebra of $\mathcal{B}(H^p)/K(H^p)$, the algebra of all bounded linear operators modulo compact operators. Thus we have a commutative Banach algebra. We identify its maximal ideal space and its Gelfand transform. For $p = 2$ our Banach algebra is a C* algebra.

In chapter 8, we use the integral representation of our composition operator and the conditions on the inducing function φ to approximate our composition operator by finite sums of multiplication and convolution operators in $\mathcal{A}_2$, thereby obtaining $C_\varphi \in \mathcal{A}_2$. We have the following main result:
Theorem A. Let $\varphi : \mathbb{H} \to \mathbb{H}$ be analytic and extend analytically across $\mathbb{R}$. Let

φ also satisfy the following:

(a) The function $\psi(z) = \varphi(z) - z$ is a bounded analytic function on $\mathbb{H}$ that is analytic also at ∞,

(b) the imaginary part of ψ satisfies $\Im(\psi(z)) \geq M > 0$ for all $z \in \mathbb{H}$ for some $M > 0$.Then for $1 < p < \infty$,

i-) $C_\varphi : H^p(\mathbb{H}) \to H^p(\mathbb{H})$ is bounded and $C_\varphi : H^2 \to H^2$ is essentially normal.

ii-) The essential spectrum of $C_\varphi : H^2(\mathbb{H}) \to H^2(\mathbb{H})$ is given by

$$\sigma_e(C_\varphi) = \overline{\{e^{i\psi(x)t} : x \in \mathbb{R}, t \in (0, \infty)\}}.$$

In chapter 9 we use the isometric isomorphism between $H^p(\mathbb{D})$ and $H^p(\mathbb{H})$ to represent the composition operator on $\mathbb{D}$ as an integral operator. As we observe in section 2, C_φ on $H^p(\mathbb{D})$ becomes a weighted composition operator on $H^p(\mathbb{H})$. We observe that the weight function is continuous at infinity. We have the following analogous result for the unit disc:

Theorem B. If $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic function of the following form

$$\varphi(w) = \frac{w + \eta(w)(1 - w)}{1 + \eta(w)(1 - w)}$$

where $\eta : \mathbb{D} \to \mathbb{C}$ is a bounded analytic function with $\Re(\eta(w)) > M > 0$ for all $w \in \mathbb{D}$ and η extends analytically across $\mathbb{T}$ then

1-) the operator $C_\varphi : H^2 \to H^2$ is essentially normal,

2-) the essential spectrum $\sigma_e(C_\varphi)$ on $H^2(\mathbb{D})$ is given by

$$\sigma_e(C_\varphi) =$$

$$\overline{\{\left(\frac{x + 2i\eta(e^{i\theta}) + i}{x + i}\right) e^{-2\eta(e^{i\theta})t} : x \in \mathbb{R}, t \in [0, \infty)\}}$$

and

$$e^{i\theta} = (x - i)/(x + i).$$

In chapter 10 we calculate the essential spectra of composition operators C_φ on $H^\infty(\mathbb{H})$, the space of bounded analytic functions of the upper half-plane for which the inducing function φ satisfies the conditions that φ is analytic across the boundary, $b(z) = \varphi(z) - z$ is a bounded analytic function on $\mathbb{H}$, the closure of the image of $\mathbb{H}$ under b is compact in $\mathbb{H}$ and $\lim_{z\to\infty} b(z) = b_0$ exists. We first characterize the spectrum and essential spectrum of T_{b_0} where $T_{b_0}f(z) = f(z+b_0)$ and then we show that $C_\varphi - T_{b_0}$ is a compact operator on $H^\infty(\mathbb{H})$. Since the essential spectrum is invariant under compact perturbations we conclude that the essential spectrum of C_φ is the same as the essential spectrum of T_{b_0}. As a corollary we also obtain the essential spectrum of C_φ on $H^\infty(\mathbb{D})$, the space of bounded analytic functions of the unit disc for which the conjugate $\tau^{-1}\varphi\tau$ of φ with respect to the Cayley transform $\tau(z) = (z-i)/(z+i)$ satisfies the above conditions. The main results of this chapter are the following:

Theorem C. Let $\varphi : \mathbb{H} \to \mathbb{H}$ be an analytic self-map of the upper half plane satisfying

(a) $\varphi(z) = z + b(z)$ where $b : \mathbb{H} \to \mathbb{H}$ is a bounded analytic function satisfying $\Im(b(z)) \geq M > 0$ for all $z \in \mathbb{H}$ and for some M positive,

(b) The limit $\lim_{z\to\infty} b(z) = b_0$ exists and $b_0 \in \mathbb{H}$.

Let $T_{b_0} : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$ be the translation operator $T_{b_0}f(z) = f(z+b_0)$. Then we have

$$\sigma_e(C_\varphi) = \sigma_e(T_{b_0}) = \{e^{itb_0} : t \in [0,\infty)\} \cup \{0\}$$

Theorem D. If $\varphi : \mathbb{D} \to \mathbb{D}$ is of the following form

$$\varphi(w) = \frac{2iw + b(\frac{i(1-w)}{1+w})(1-w)}{2i + b(\frac{i(1-w)}{1+w})(1-w)}$$

with $b : \mathbb{H} \to \mathbb{H}$ bounded analytic with $b(\mathbb{H}) \subset\subset \mathbb{H}$ and $\lim_{z\to\infty} b(z) = b_0$ then for $C_\varphi : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ we have

$$\sigma_e(C_\varphi) = \{e^{itb_0} : t \in [0,\infty)\} \cup \{0\}.$$

CHAPTER 2

HARDY SPACES

The Hardy space of the unit disc is defined to be the set of analytic functions g on $\mathbb{D}$ for which there is a $C > 0$ satisfying

$$\int_{-\pi}^{+\pi} |g(re^{i\theta})|^p d\theta < C, \qquad 0 < r < 1.$$

For $1 \leq p < \infty$, $H^p(\mathbb{D})$ is a Banach space with norm defined by

$$\| g \|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{+\pi} |g(re^{i\theta})|^p d\theta, \qquad g \in H^p(\mathbb{D}).$$

We will always think of $H^p(\mathbb{D})$ as embedded in $L^p(\mathbb{T})$ via the embedding $g \longrightarrow g^*$, where $g^*(e^{i\theta}) = \lim_{r\to 1} g(re^{i\theta})$ is the radial boundary value function of g.

The Hardy space $H^p(\mathbb{H})$ of the upper half plane $\mathbb{H}$ is defined to be the set of all analytic functions f on $\mathbb{H}$ for which there exists $C > 0$ such that

$$\int_{-\infty}^{+\infty} |f(x+iy)|^p dx < C, \qquad y > 0.$$

For $1 \leq p < \infty$, $H^p(\mathbb{H})$ is a Banach space with norm defined by

$$\| f \|_p^p = \sup_{0<y<\infty} \int_{-\infty}^{+\infty} |f(x+iy)|^p dx.$$

In a similar manner as done in the unit disc case one imbeds $H^p(\mathbb{H})$ in $L^p(\mathbb{R})$ via $f \longrightarrow f^*$ where $f^*(x) = \lim_{y\to 0} f(x+iy)$. This embedding is an isometry.

The two Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{H})$ are isometrically isomorphic. An isomorphism $\Psi : H^p(\mathbb{D}) \longrightarrow H^p(\mathbb{H})$ is given by

$$\Psi(g)(z) = \left(\frac{1}{\sqrt{\pi}(z+i)}\right)^{\frac{2}{p}} g\left(\frac{z-i}{z+i}\right)$$

We claim that the operator Ψ is an isometry, to see this let $g \in H^p(\mathbb{D})$ and let $e^{i\theta} = (x-i)/(x+i)$ then for $z = x+it$ we have for a.a. $\theta \in [0, 2\pi)$

$$\lim_{t \to 0} g\left(\frac{x+it-i}{x+it+i}\right) = g^*(e^{i\theta}).$$

So

$$\Psi(g)^*(x) = \lim_{t \to 0} \Psi(g)(x+it) = \left(\frac{1}{\sqrt{\pi}(x+i)}\right)^{\frac{2}{p}} g^*(e^{i\theta}).$$

Taking into account that $d\theta = \frac{2dx}{1+x^2}$ we obtain

$$\| \Psi(g)(x) \|_p^p = \int_{-\infty}^{\infty} | \Psi(g)^*(x) |^p \, dx = \int_{-\infty}^{\infty} | \frac{1}{\pi^{\frac{1}{2}}(x+i)} |^2 | \, g^*\left(\frac{x-i}{x+i}\right) |^p \, dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} | \, g^*\left(\frac{x-i}{x+i}\right) |^p \, \frac{dx}{1+x^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} | \, g^*(e^{i\theta}) |^p \, d\theta = \| g \|_p^p .$$

Thus the operator Ψ is an isometry as asserted. The mapping Ψ is onto and invertible with inverse $\Phi : H^p(\mathbb{H}) \longrightarrow H^p(\mathbb{D})$ given by

$$\Phi(f)(z) = \frac{(1-z)^{\frac{2}{p}}}{2\pi^{\frac{1}{p}} i} f\left(\frac{i(1+z)}{1-z}\right).$$

We have $\Psi(\Phi(f)) = f$ for all $f \in H^p(\mathbb{H})$ and $\Phi(\Psi(g)) = g$ for all $g \in H^p(\mathbb{D})$. For more details see [15, pp. 128-131].

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic self-map of the unit disc. The composition operator

$$C_\varphi : H^p(\mathbb{D}) \longrightarrow H^p(\mathbb{D})$$

is defined by

$$C_\varphi(g)(z) = g(\varphi(z)), \qquad z \in \mathbb{D}.$$

Similarly for an analytic selfmap ψ of the upper half-plane, the composition operator

$$C_\psi : H^p(\mathbb{H}) \longrightarrow H^p(\mathbb{H})$$

is defined by

$$(C_\psi f)(z) = f(\psi(z)), \qquad z \in \mathbb{H}.$$

Composition operators of the unit disc are always bounded [7] whereas composition operators of the upper half-plane are not always bounded [18]. For more information on composition operators of the unit disc see [7]. For the boundedness problem of composition operators of the upper half-plane see [18].

The Cayley transform $\tau(z) = (z-i)/(z+i)$ maps the upper half plane conformally onto the unit disc. The composition operator C_φ on $H^p(\mathbb{D})$ is carried over to $(\frac{\tilde{\varphi}(z)+i}{z+i})^{\frac{2}{p}} C_{\tilde{\varphi}}$ on $H^p(\mathbb{H})$ through Ψ where $\tilde{\varphi} = \tau\varphi\tau^{-1}$ i.e. we have

$$\Psi C_\varphi \Phi = \left(\frac{\tilde{\varphi}(z)+i}{z+i}\right)^{\frac{2}{p}} C_{\tilde{\varphi}}.$$

Throughout we will identify composition operators of the unit disc with weighted composition operators of the upper half-plane. So we will use the term composition operator to indicate the composition operator of the upper half-plane.

CHAPTER 3

THE CAUCHY KERNEL

In order to represent our composition operator with an integral kernel we first observe that any function in $H^p(\mathbb{H})$ can be recovered from its boundary values by means of the Cauchy integral. To prove this we first prove the following lemma from [16 pp. 149].

Lemma 3.1 If $f \in H^p(\mathbb{H})$, where $1 \leq p < +\infty$, then

$$| f(z) | \leq \frac{2^{\frac{1}{p}}}{(\pi \Im(z))^{\frac{1}{p}}} \parallel f \parallel_p, \qquad z \in \mathbb{H}.$$

Proof. The function $z \to | f(z) |^p$ is subharmonic so for any $z = x + iy \in \mathbb{H}$

$$| f(z) |^p \leq \frac{1}{2\pi r} \int_0^{2\pi} | f(z + re^{i\theta}) |^p \, d\theta, 0 < r < y.$$

Multiplying by r and integrating from 0 to y, we obtain

$$\frac{y^2}{2} | f(z) |^p \leq \frac{1}{2\pi} \int_0^y \int_0^{2\pi} | f(z + re^{i\theta}) |^p \, r dr d\theta$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{2y} | f(\xi + i\eta) |^p \, d\xi d\eta \leq \frac{y}{\pi} \parallel f \parallel_p^p .$$

Since $y = \Im(z)$ the lemma is proved. $\square$

As a result we have the following theorem:

Theorem 3.2 Let $f \in H^p(\mathbb{H})$, $1 \leq p < \infty$, and let f^* be its nontangential boundary value function on $\mathbb{R}$. Then

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f^*(x) dx}{x - z}, \qquad z \in \mathbb{H}.$$

Proof. Take $0 < h < \Im(z)$. Let $\Gamma_1 = \{x + ih : -R\cos(\theta_0) \leq x \leq R\cos(\theta_0)\}$, $\Gamma_2 = \{Re^{i\theta} : \theta_0 < \theta < \pi - \theta_0\}$, $h = R\sin(\theta_0)$ and $\Gamma_R = \Gamma_1 \cup \Gamma_2$. Then by the Cauchy integral formula we have the following

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(\xi)d\xi}{\xi - z}.$$

We split the left hand side as follows

$$\frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(\xi)d\xi}{\xi - z} = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\xi)d\xi}{\xi - z} + \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\xi)d\xi}{\xi - z}$$

$$= \frac{1}{2\pi i} \int_{-R\cos(\theta_0)}^{R\cos(\theta_0)} \frac{f(\xi + ih)d\xi}{\xi - z} + \frac{1}{2\pi i} \int_{\theta_0}^{\pi-\theta_0} \frac{f(Re^{i\theta_0})iRe^{i\theta}d\theta}{Re^{i\theta} - z}$$

Now we will show that the second integral converges to 0 as $R \to +\infty$: by the above lemma we have

$$| f(Re^{i\theta}) | \leq \frac{2^{\frac{1}{p}} \| f \|_p}{\pi^{\frac{1}{p}}(R\sin\theta)^{\frac{1}{p}}}$$

then we have for $p > 1$

$$| \frac{1}{2\pi i} \int_{\theta_0}^{\pi-\theta_0} \frac{f(Re^{i\theta_0})iRe^{i\theta}d\theta}{Re^{i\theta} - z} | \leq \frac{1}{2\pi} \int_{\theta_0}^{\pi-\theta_0} \frac{| f(Re^{i\theta}) | Rd\theta}{| Re^{i\theta} - z |}$$

$$\leq \frac{C}{2\pi R^{\frac{1}{p}}} \frac{R}{R- | z |} \int_0^{\pi} \frac{d\theta}{(\sin(\theta))^{\frac{1}{p}}} \longrightarrow 0$$

as $R \to +\infty$. For $p = 1$ we use the fact that $\sin(\theta) \geq C_0\theta \ \forall\theta \in (\theta_0, \frac{\pi}{2}]$ for some $C_0 > 0$ and $\sin(\theta) \geq C_0(\pi - \theta) \ \forall\theta \in [\frac{\pi}{2}, \pi - \theta_0)$ to get

$$\frac{1}{R} \int_{\theta_0}^{\pi-\theta_0} \frac{d\theta}{(\sin(\theta))} \leq \frac{2}{C_0R} \int_{\theta_0}^{\frac{\pi}{2}} \frac{d\theta}{\theta} = \frac{2}{C_0R} \log \frac{\pi}{2\theta_0} \leq \frac{2}{C_0R} \log \frac{\pi}{2\sin(\theta_0)}$$

$$= \frac{2}{C_0R} \log \frac{\pi R}{2h} \longrightarrow 0$$

as $R \to +\infty$. Hence we have

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(x+ih)dx}{x-z} \qquad 0 < h < \Im(z)$$

Let $G_h(x) = \frac{f(x+ih)}{x-z}$ then $\mid G_h(x) \mid < \frac{C}{\Im(z)}$ for some $C > 0$ and $G_h(x) \to \frac{f^*(x)}{x-z} = G(x)$ as $h \to 0$ for a.e $x \in \mathbb{R}$.So by Lebesgue's dominated convergence theorem

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} G_h(x)dx \to \frac{1}{2\pi i}\int_{-\infty}^{+\infty} G(x)dx.$$

And this proves our theorem $\square$

One may get the integral formula in the theorem for $p = 2$ by directly appealing to Lemma 3.1 above. Because Lemma 3.1 extracts one of the most important properties of the Hilbert space $H^2(\mathbb{H})$, namely the reproducing kernel property. We will use this property of $H^2(\mathbb{H})$ to get the integral formula in Theorem 3.2. for $p = 2$.

Let H be a Hilbert space of analytic functions on a domain Ω. To every $z \in \Omega$ a linear functional δ_z called the point evaluation at z, is attached such that $\delta_z : H \to \mathbb{C}$, $\delta_z(f) = f(z)$. The Hilbert space H is called a *reproducing kernel Hilbert space* (RKHS for short)if for all $z \in \Omega$ the point evaluation δ_z, is bounded on H. This definition was introduced by N. Aronszjan in (Aronszajn, N. Theory of reproducing kernels. Trans. Amer. Math. Soc. 68, (1950)). Since H is its own dual by Riesz representation theorem there exists a unique $g_z \in H$ such that $\delta_z(f) = < f, g_z >$ for all $f \in H$. The function g_z is called the reproducing kernel. By Lemma 3.1. $H^2(\mathbb{H})$ is a reproducing kernel Hilbert space so it is a reasonable task to compute its reproducing kernel

To compute g_z in our case we take an orthonormal basis of $H^2(\mathbb{H})$. Let e_n be defined as

$$e_n(w) = \frac{1}{\pi^{\frac{1}{2}}(w+i)}\left(\frac{w-i}{w+i}\right)^n.$$

Referring to the the isometric equivalence of $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$ we see that $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis of $H^2(\mathbb{H})$ (This bases corresponds to $\{z^n\}$ in

$H^2(\mathbb{D})$ which is an orthonormal bases for $H^2(\mathbb{D})$). Hence

$$g_z(w) = \sum_{n=0}^{\infty} < g_z, e_n > e_n(w).$$

And we have

$$< g_z, e_n >= \overline{\delta_z(e_n)} = \overline{\frac{1}{\pi^{\frac{1}{2}}} \left(\frac{1}{z+i} \right) \left(\frac{z-i}{z+i} \right)^n}$$

So we have

$$g_z(w) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{\overline{(z+i)}} \overline{\left(\frac{z-i}{z+i} \right)^n} \frac{1}{(w+i)} \left(\frac{w-i}{w+i} \right)^n$$

$$= \frac{1}{\pi \overline{(z+i)}(w+i)} \frac{1}{1 - \overline{(\frac{z-i}{z+i})}(\frac{w-i}{w+i})} = \frac{1}{2\pi i(\overline{z} - w)}$$

So the reproducing kernel g_z for $H^2(\mathbb{H})$ is the function

$$g_z(w) = \frac{1}{2\pi i(\overline{z} - w)}.$$

So one has

$$f(z) = \delta_z(f) =< f, g_z >= \int_{-\infty}^{+\infty} f^*(x)\overline{g_z^*(x)}dx = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f^*(x)dx}{x - z}.$$

Therefore for any $f \in H^2(\mathbb{H})$ and $z \in \mathbb{H}$ we have

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f^*(x)dx}{x - z}.$$

One can use the Cauchy integral formula (Theorem 3.2) to represent composition operators with an integral kernel under some conditions on the analytic symbol $\varphi : \mathbb{H} \to \mathbb{H}$. By Fatou's theorem on the boundary values of the $H^\infty(\mathbb{D})$ functions one may deduce that (going back and forth with Cayley transform) for any analytic function $\varphi : \mathbb{H} \to \mathbb{H}$ the limit $\lim_{t \to 0} \varphi(x + it) = \varphi^*(x)$ exists for a.a. $x \in \mathbb{R}$. This will be of minor importance since we will work with φ that extends continuously to $\mathbb{R}$. The most important condition that we will pose on

φ is $\Im(\varphi^*(x)) > 0$ for a.a. $x \in \mathbb{R}$ where $\varphi^*(x) = \lim_{t\to 0} \varphi(x+it)$. By the integral formula above one has

$$C_\varphi(f)(x+it) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f^*(\xi)d\xi}{\xi - \varphi(x+it)}.$$

Let $x \in \mathbb{R}$ be s.t. $\lim_{t\to 0}\varphi(x+it) = \varphi^*(x)$ exists and $\Im(\varphi^*(x)) > 0$ then consider

$$\mid C_\varphi(f)(x+it) - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f^*(\xi)d\xi}{\xi-\varphi^*(x)} \mid$$

$$=\mid \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f^*(\xi)d\xi}{\xi-\varphi(x+it)} - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f^*(\xi)d\xi}{\xi-\varphi^*(x)} \mid$$

$$= \frac{1}{2\pi}\mid \varphi(x+it) - \varphi^*(x) \mid\mid \int_{-\infty}^{\infty}\frac{f^*(\xi)d\xi}{(\xi-\varphi(x+it))(\xi-\varphi^*(x))} \mid$$

$$\leq \frac{\mid \varphi(x+it)-\varphi^*(x)\mid}{2\pi} \parallel f \parallel_p \left(\int_{-\infty}^{\infty}\frac{d\xi}{(\mid (\xi-\varphi(x+it))(\xi-\varphi^*(x)) \mid)^q}\right)^{\frac{1}{q}}$$

by Hölder's inequality. When $\mid \varphi(x+it) - \varphi^*(x) \mid < \varepsilon$ we have $\mid \xi - \varphi(x+it) \mid \geq \mid \xi - \varphi^*(x) \mid - \varepsilon$

Fix $\varepsilon_0 > 0$ s.t. $\mid \xi - \varphi^*(x) \mid > \varepsilon_0$ for all $\xi \in \mathbb{R}$. Since $\Im(\varphi^*(x) > 0$ this is possible. Let $\varepsilon > 0$ s.t. $\varepsilon_0 > \varepsilon$ then since $\lim_{t\to 0}\varphi(x+it) = \varphi^*(x)$ exists, there exists $\delta > 0$ s.t. for all $0 < t < \delta$ we have $\mid \varphi(x+it) - \varphi^*(x) \mid < \varepsilon < \varepsilon_0$. So one has

$$\mid \xi - \varphi(x+it) \mid \geq \mid \xi - \varphi^*(x) \mid - \varepsilon_0$$

for all t s.t. $0 < t < \delta$

$$\mid C_\varphi(f)(x+it) - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f^*(\xi)d\xi}{\xi-\varphi^*(x)} \mid$$

$$\leq \frac{\mid \varphi(x+it)-\varphi^*(x)\mid}{2\pi} \parallel f \parallel_p \left(\int_{-\infty}^{\infty}\frac{d\xi}{\mid \xi-\varphi^*(x)\mid^{2q} - \varepsilon_0 \mid \xi - \varphi^*(x) \mid^q}\right)^{\frac{1}{q}}$$

$$= \frac{\mid \varphi(x+it)-\varphi^*(x)\mid}{2\pi} \parallel f \parallel_p M_{\varepsilon_0,x} \leq \frac{\varepsilon}{2\pi} \parallel f \parallel_p M_{\varepsilon_0,x}$$

where $M_{\varepsilon_0,x} = (\int_{-\infty}^{\infty} \frac{d\xi}{|\xi-\varphi^*(x)|^{2q}-\varepsilon_0|\xi-\varphi^*(x)|^q})^{\frac{1}{q}}$. Hence we have

$$\lim_{t\to 0} C_\varphi(f)(x+it) = C_\varphi(f)^*(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f^*(\xi)d\xi}{\xi-\varphi^*(x)}$$

for a.e. $x \in \mathbb{R}$

We summarize the result of our discussion in the following theorem:

Theorem 3.3 Let $\varphi : \mathbb{H} \to \mathbb{H}$ be an analytic function such that the nontangential boundary value function φ^* of φ satisfies $\Im(\varphi^*(x)) > 0$ for almost all $x \in \mathbb{R}$. Then the composition operator C_φ on $H^p(\mathbb{H})$ for $1 < p < \infty$ is given by

$$C_\varphi(f)^*(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f^*(\xi)d\xi}{\xi-\varphi^*(x)} \quad \text{for a.a.} \quad x \in \mathbb{R}.$$

CHAPTER 4

THE PALEY WIENER THEOREM

The Fourier transform $\mathcal{F}f$ of $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$(\mathcal{F}f)(t) = \hat{f}(t) = \int_{-\infty}^{+\infty} e^{-2\pi itx} f(x)dx.$$

The Fourier transform extends to an invertible isometry from $L^2(\mathbb{R})$ onto itself with inverse

$$(\mathcal{F}^{-1}f)(t) = \check{f}(t) = \int_{-\infty}^{+\infty} e^{2\pi itx} f(x)dx$$

Moreover $\mathcal{F}$ extends to $L^1(\mathbb{R})$ and maps $L^1(\mathbb{R})$ boundedly into $L^\infty(\mathbb{R})$. In fact the image of $L^1(\mathbb{R})$ under $\mathcal{F}$ falls into $C_0(\mathbb{R})$, the space of continuous functions vanishing at infinity. Since $\mathcal{F}$ maps L^1 boundedly into L^∞ and maps L^2 isometrically onto itself, by the Riesz-Thorin interpolation theorem, it maps L^p boundedly into L^q where $1 < p \leq 2$ and $p = q/(q-1)$. For $2 < p \leq \infty$ $\mathcal{F}$ does not extend boundedly to $L^p(\mathbb{R})$. The first chapter of [22] gives a brief treatment of such mapping properties of the Fourier transform. Another elementary source about this subject is [23].

The Hilbert transform $\mathcal{H}$ is the singular integral operator defined by

$$(\mathcal{H}f)(x) = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}dy \qquad x \in \mathbb{R}$$

For $1 < p < \infty$, the operator $\mathcal{H}$ is bounded on $L^p(\mathbb{R})$ i.e.

$$\| \mathcal{H}f \|_p \leq C_p \| f \|_p \qquad 1 < p < \infty$$

where C_p only depends on p.Let S be the Cauchy singular integral operator defined as follows

$$S(f)(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(w)}{w-z}dw \qquad z \in \mathbb{H}.$$

Recall that for any $f \in L^p(\mathbb{R})$

$$f \in H^p \quad \text{if and only if} \quad \lim_{t\to 0} S(f)(x+it) = f(x)$$

for a.e. $x \in \mathbb{R}$. Observe that S can be split up as follows

$$S(f)(z) = \frac{1}{2}((P_t * f)(x) + i(Q_t * f)(x))$$

for $z = x + it \in \mathbb{H}$ where $P_t(x) = \frac{1}{\pi}(\frac{t}{x^2+t^2})$ is the Poisson kernel and $Q_t(x) = \frac{1}{\pi}(\frac{x}{x^2+t^2})$ is the conjugate Poisson kernel. Since the Poisson kernel is an approximate identity on L^p we observe that for any $f \in L^p$

$$\lim_{t\to 0}(P_t * f)(x) = f(x)$$

for a.e. $x \in \mathbb{R}$. We also observe the following intimate relation between the conjugate Poisson kernel and the Hilbert transform

$$\lim_{t\to 0}(Q_t * f)(x) = \mathcal{H}f(x)$$

for a.e. $x \in \mathbb{R}$. Summing all these up we have

$$f \in H^p \quad \text{if and only if} \quad (I - i\mathcal{H})f = 0$$

If we take the Fourier transform of $\mathcal{H}(f)$ we obtain

$$(\mathcal{H}f)\hat{}(x) = -i \quad \operatorname{sgn}(x)\hat{f}(x)$$

where, $\operatorname{sgn}(x) = 1$ if $x \in \mathbb{R}$ is positive and $\operatorname{sgn}(x) = -1$ if $x \in \mathbb{R}$ is negative, is the signum function. And then we have

$$(I - i\mathcal{H})(f)\check{}(x) = 2\chi_{(-\infty,0]}\hat{f}(x)$$

where $\chi_{(-\infty,0]}$ is the characteristic function of $(-\infty, 0]$. Since the Fourier transform is injective, gathering all these we have the following theorem:

Theorem 4.1 Fix $1 < p < \infty$. For $f \in L^p(\mathbb{R})$, the following assertions are equivalent;

(i) $f \in H^p$,

(ii) $(I - i\mathcal{H})f = 0$,

(iii) $\mathrm{supp}(\hat{f}) \subseteq [0, \infty)$.

The special case $p = 2$ of this theorem was discovered by R.E.A.C. Paley and N. Wiener in 1933. Because of this it is referred to as the Paley-Wiener theorem. See [11 pp.88] and [12 pp.110-111].

CHAPTER 5

TRANSLATION SEMIGROUP ON THE UPPER HALF-PLANE

In this chapter we will analyze the composition operator induced by translations on the Hardy spaces of the upper halfplane. More precisely it is the composition operator $C_\varphi : H^p(\mathbb{H}) \to H^p(\mathbb{H})$ where $\varphi(z) = z + \alpha$ for some $\alpha \in \mathbb{H}$. For $p = 2$, as we will show below, these operators are unitarily similar to multiplication operators. For other p values they intertwine with multiplication operators. They have been investigated by W. Higdon in [13] and E.A. Gallardo-Gutierrez and A. Montes-Rodriguez in [9].

Higdon, in his analysis of translations, uses the strongly continuous semigroup property of these operators. Gallardo-Gutierrez and Montes-Rodriguez use Paley Wiener Theorem together with the identity

$$(\mathcal{F}T_a f)(t) = e^{iat}(\mathcal{F}f)(t), \qquad t \in \mathbb{R}$$

where $T_a(f)(z) = f(z+a)$. Here we take a slightly different approach. Instead of the Fourier transform we will use the Cauchy kernel, which will eventually lead to the opportunity to analyze a larger class of composition operators.

Consider $\varphi : \mathbb{H} \longrightarrow \mathbb{H}$, $\varphi(z) = z + \alpha$ where $\alpha \in \mathbb{C}$, $\Im(\alpha) > 0$. We will analyze the spectrum and the essential spectrum of the operator $C_\varphi : H^2(\mathbb{H}) \to H^2(\mathbb{H})$. Set $f^*(x) = \lim_{y \to 0} f(x + iy)$. By Theorem 3.2 we have

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f^*(t)}{t - z}, \qquad z \in \mathbb{H}, f \in H^2(\mathbb{H})$$

Substituting $z+\alpha$ for z, we obtain

$$f(z+\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f^*(t)}{t-z-\alpha}dt.$$

Since $\Im(\alpha) > 0$, we have $C_\varphi(f)^*(x) = f(x+\alpha)$. Let

$$k(x) = \frac{1}{2\pi i}\frac{-1}{x+\alpha},$$

thus we have

$$C_\varphi(f)^*(x) = f(x+\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f^*(t)}{t-x-\alpha}dt = (k * f)^*(x)$$

So the imbedding $f \to f^*$ of $H^2(\mathbb{H})$ into $L^2(\mathbb{R})$ makes C_φ an integral operator of convolution type.

The Paley-Wiener theorem asserts that the image of $H^2(\mathbb{H})$ under $\mathcal{F}$ is $L^2(0,\infty)$. And by the Fourier inversion theorem $\mathcal{F}$ is invertible on $L^2(\mathbb{R})$. Consider

$$\mathcal{F}C_\varphi\mathcal{F}^{-1} : L^2([0,\infty)) \to L^2([0,\infty))$$

then since $\mathcal{F}$ converts convolutions to ordinary multiplication we have

$$\mathcal{F}C_\varphi\mathcal{F}^{-1}(g) = \hat{k}g.$$

Let $a \in L^\infty(\mathbb{R})$, the multiplication operator M_a by a on $L^2([0,\infty))$ is defined to be

$$M_a(f)(x) = a(x)f(x).$$

Since a is essentially bounded M_a is a bounded operator from $L^2([0,\infty))$ into itself. If a is continuous i.e. $a \in C([0,\infty))$ with $\lim_{t\to\infty} a(t) = a_0$ exists then the spectrum and the essential spectrum of M_a coincide and they are given by

$$\sigma(M_a) = \sigma_e(M_a) = \overline{a(0,\infty)}.$$

For more information on this we refer the reader to [19].

So C_φ is transformed into a multiplication operator on $L^2(0,\infty)$. Since the Fourier transform is invertible

$$\sigma(C_\varphi) = \sigma(\mathcal{F}C_\varphi\mathcal{F}^{-1}) = \sigma(M_{\hat{k}})$$

and

$$\sigma_e(C_\varphi) = \sigma_e(\mathcal{F}C_\varphi\mathcal{F}^{-1}) = \sigma_e(M_{\hat{k}})$$

So it remains for us to compute the Fourier transform of k which is

$$\hat{k}(t) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-itx}}{x+\alpha}dx$$

By complex contour integration we obtain

$$\hat{k}(t) = e^{i\alpha t}$$

We observe that $\hat{k} \in C([0,\infty))$ with $\lim_{t\to\infty} \hat{k}(t) = 0$ so we have

$$\sigma(M_a) = \sigma_e(M_a) = \overline{a(0,\infty)}$$

and as a result we have

$$\sigma(M_{\hat{k}}) = \sigma_e(M_{\hat{k}}) = \{e^{i\alpha t} : 0 \le t < \infty\} \cup \{0\}.$$

Hence the spectrum of C_φ operating on $H^2(\mathbb{H})$ is given by

$$\sigma(C_\varphi) = \sigma_e(C_\varphi) = \sigma(M_{\hat{k}}) = \sigma_e(M_{\hat{k}}) = \{e^{i\alpha t} : 0 \le t < \infty\} \cup \{0\},$$

which is a spiral curve starting at 1 and spiraling to 0.

More generally we consider the algebra of convolution operators and find the spectra of our operator in the algebra of convolution operators in the following way: Define $\mathcal{C}$ to be the algebra of operators K on H^p of the form

$$Kf = \lambda f + k * f$$

for some $k \in L^1$ and $\lambda \in \mathbb{C}$. Convolution operators map H^p into H^p by the Paley-Wiener theorem and by the fact that Fourier transform converts convolution to multiplication i.e. let $k \in L^1(\mathbb{R})$ and take $f \in H^p$ then by Minkowski-Young inequality [22] $k * f \in L^p(\mathbb{R})$. Consider $(k * f)\hat{}$, since

$$(k * f)\hat{}\ \ (t) = \hat{k}(t)\hat{f}(t)$$

and $\text{supp}(\hat{f}) \subseteq [0, \infty)$ we have

$$\text{supp}((k * f)\hat{}\ \ ) \subseteq [0, \infty).$$

Hence by Paley-Wiener theorem $k * f \in H^p$. Now let $\mathcal{C}_p$ be the closure of $\mathcal{C}$ in the operator norm, i.e. in $\mathcal{B}(H^p)$. Then $\mathcal{C}_p$ is a commutative Banach algebra with identity. Its maximal ideal space is $[0, \infty]$ and the Gelfand transform coincides with the Fourier transform. It can be easily seen that, $\mathcal{C}_2$ preserves the adjoint: For any $K \in \mathcal{C}_2$ in the following form

$$K(f)(x) = \lambda f(x) + k * f(x)$$

for $k \in L^1$ one can easily observe that

$$K^* f(x) = \overline{\lambda} f(x) + \tilde{k} * f(x)$$

where $\tilde{k}(x) = \overline{k(-x)}$. So for $p = 2$ $\mathcal{C}_2$ is a C* algebra and hence by Gelfand-Naimark theorem it is isometrically isomorphic to $C[0, \infty]$ the algebra of continuous functions on $[0, \infty]$, where the topology of $[0, \infty]$ is the one induced by the one point compactification $\tilde{\mathbb{R}}$ of $\mathbb{R}$. Since $\mathcal{C}_2$ is a C* subalgebra of $\mathcal{B}(H^2)$, for any $K \in \mathcal{C}_2$ with $K(f)(x) = k * f(x)$ for some $k \in L^1$ we have

$$\sigma(K) = \sigma_{\mathcal{C}_2}(K) = \sigma_{C[0,\infty]}(\hat{k}) = \hat{k}([0, \infty]).$$

As for the essential spectrum we construct a singular sequence for any $\lambda \in \sigma(C_\varphi)$ i.e a sequence of functions $\{g_k\}_{k=1}^{\infty}$ such that $\frac{g_k}{\|g_k\|_2}$ has no convergent subsequence in H^2 and

$$\frac{\| C_\varphi g_k - \lambda g_k \|_2}{\| g_k \|_2} \to 0$$

as $k \to \infty$. Now take $\lambda = e^{2\pi i t_0 \alpha}$ for $t_0 > 0$. Fix a sequence $\varepsilon_k \in \mathbb{R}$ such that $\varepsilon_k > 0$, $\varepsilon_{k+1} < \varepsilon_k$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \varepsilon_k = 0$. Consider

$$g_k(x) = \int_{t_0+\varepsilon_{k+1}}^{t_0+\varepsilon_k} e^{2\pi i x t} dt.$$

We observe that $g_k \in L^2(\mathbb{R})$ and g_k's satisfy

$$\hat{g_k} = \chi_{(t_0+\varepsilon_{k+1}, t_0+\varepsilon_k)}$$

and they are mutually orthogonal since

$$\int g_i \overline{g_j} = \int \hat{g_i} \overline{\hat{g_j}} = (\varepsilon_i - \varepsilon_{i+1})\delta_{ij}$$

where δ_{ij} is the Kroenecker delta. Since they are mutually orthogonal $\frac{g_k}{\|g_k\|_2}$ has no convergent subsequence in H^2. Using the Mean Value Theorem we have the following estimate for g_k's:

$$\| C_\varphi g_k - e^{2\pi i t_0 \alpha} g_k \|_2 \quad \leq \quad | 2\pi\alpha | \, \varepsilon_{k+1} \| g_k \|_2 \, .$$

Hence we have

$$\frac{\| C_\varphi g_k - e^{2\pi i t_0 \alpha} g_k \|_2}{\| g_k \|_2} \to 0$$

as $k \to \infty$. This means that $\lambda = e^{2\pi i t_0 \alpha} \in \sigma_e(C_\varphi)$ since had we $\lambda \notin \sigma_e(C_\varphi)$, $\lambda I - C_\varphi$ would be invertible in the Calkin algebra i.e $\exists\, T \in \mathcal{B}(H^2)$ and $K \in K(H^2)$ such that

$$T(\lambda I - C_\varphi) = I + K.$$

Since K is a compact operator we have $K \frac{g_k}{\|g_k\|_2} \to 0$ in H^2 and this leads to a contradiction. As a result we have for $\alpha \in \mathbb{H}$ and $\varphi(z) = z + \alpha$, $C_\varphi : H^2(\mathbb{H}) \to H^2(\mathbb{H})$

is a bounded linear operator with spectrum and essential spectrum satisfying

$$\sigma(C_\varphi) = \sigma_e(C_\varphi) = \{e^{i\alpha t} : 0 \leq t < \infty\} \cup \{0\}.$$

In the example above our operator is transformed into an operator of convolution type is due to the fact that $\varphi(x) - x = \alpha$ is a constant. We wish to extend the method we described above to more general settings considering the kernel $\frac{1}{2\pi i(\varphi(x)-y)} = k(x, x-y)$ of our composition operator as a variable kernel where $k(x,z) = \frac{1}{2\pi i(\varphi(x)-x+z)}$.

CHAPTER 6

ALGEBRAS WITH SYMBOLS

The concept of "symbol" of singular integral operators was introduced by Mihlin in the beginnings of the twentieth century. He observed that an algebra of singular integral operators satisfying certain conditions on $L^2(\mathbb{R}^n)$ modulo compact operators can be put in correspondence with some algebra of continuous functions using the Fourier transform, in a one to one and onto fashion. Moreover this correspondence is an isomorphism and preserves the norms. His theory was generalized to operators on $L^p(\mathbb{R}^n)$ for arbitrary $1 < p < \infty$ by Calderón and Zygmund in [3] and [4]. The "symbol" usually coincides with the Gelfand transform.

Observing these facts Cordes and Herman in [6] introduced an abstract notion of Banach algebra with symbol along the following lines:

Let $\mathcal{B}$ be a C* algebra with identity, let A_1 and A_2 be two closed commutative *- subalgebras with identity and let $\mathcal{I}$ be a two sided closed ideal of $\mathcal{B}$ satisfying the following two conditions

(i) $ab - ba \in \mathcal{I} \qquad a \in A_1,\, b \in A_2$

(ii) $A_1 \cap \mathcal{I} = \{0\}$.

Then the closure $\mathcal{A}$ of the subalgebra generated by the linear span of A_1, A_2 and $\mathcal{I}$ is called a Banach algebra with symbol. The quotient algebra $\mathcal{A}/\,\mathcal{I}$ is a commutative C* algebra and its Gelfand transform is called the symbol.

In this setup $\mathcal{B}$ is considered to be $\mathcal{B}(L^2(\mathbb{R}^n))$ the algebra of all bounded linear operators on the Hilbert space $L^2(\mathbb{R}^n)$, $\mathcal{I}$ the ideal of compact operators in $\mathcal{B}(L^2(\mathbb{R}^n))$, A_1 the Banach algebra of multiplication operators by complex-valued continuous functions on the one point compactification of $\mathbb{R}^n$, and A_2 the Banach algebra of operators generated by the identity and L^1 convolution operators.

Since our operators act on H^2 rather than L^2, we need to modify the definition a little bit. Our algebra with symbol definition is as follows:

Definition 6.1 Let $\mathcal{B}$ be a C* algebra with identity, $\mathcal{I}$ be a closed two sided ideal of $\mathcal{B}$ and A_1, A_2 be * subalgebras with identity of $\mathcal{B}$ s.t.

(i) The subalgebra A_1 is commutative,

(ii) $A_1 \cap \mathcal{I} = \{0\}$,

(iii) $a_1 a_2 - a_2 a_1 \in \mathcal{I} \qquad a_1, a_2 \in A_1 \cup A_2$.

Then the closed subalgebra $\mathcal{A}$ of $\mathcal{B}$ generated by A_1, A_2 and $\mathcal{I}$ is called algebra with symbol.

The algebra $\mathcal{A}/\,\mathcal{I}$ is a commutative Banach algebra with identity. We will show that it is also C* algebra, hence by Gelfand-Naimark theorem $\mathcal{A}/\mathcal{I} \cong C(X)$ where X is the maximal ideal space of $\mathcal{A}/\,\mathcal{I}$.

We are interested in the case that $\mathcal{B} = \mathcal{B}(H^2)$ is the C* algebra of all bounded linear operators on the Hilbert space H^2, $\mathcal{I}$ is the ideal of compact operators, A_1 is the closure of the algebra generated by identity and L^1 convolution operators, and A_2 is the Banach algebra of operators on H^2 generated by the linear span of I, P, and P^* where I is the identity, P is the operator of multiplication by $(x-i)/(x+i)$ and P^* is the adjoint of P.

In this section we will give a proof of the following theorem taken from [8 pp.124]. We will use this result in Section 7.

Theorem 6.2 If $\mathcal{U}$ is a C* algebra and $\mathcal{I}$ is a two sided closed ideal in $\mathcal{U}$, then $\mathcal{I}$ is self-adjoint i.e $T^* \in \mathcal{I}$ whenever $T \in \mathcal{I}$ and the quotient algebra $\mathcal{U}/\mathcal{I}$ is a C* algebra with respect to the involution induced by the natural map.

We take the proof of Theorem 6.2 from [2, pp.10-12]. We first prove the following lemma that we will use in the proof of theorem 6.2.

Lemma 6.3 Let A be a C* algebra and let $x \in A$. Then there is a sequence e_1,e_2,... of self-adjoint elements such that $\sigma(e_n) \subset [0,1]$ for all n and $xe_n \to x$ as $n \to \infty$

Proof. Consider $z = x^*x$ then z is self-adjoint, so by functional calculus define

$$e_n = nz^2(e + nz^2)^{-1}.$$

Since z is self-adjoint, e_n's are also self-adjoint. And by spectral mapping theorem $\sigma(e_n) = f_n(\sigma(z))$ where $f_n(t) = nt^2(1+nt^2)^{-1}$ hence $\sigma(e_n) \subset [0,1]$. Now consider

$$e - e_n = e - nz^2(e + nz^2)^{-1} = (e + nz^2)^{-1}$$

and since $e - e_n$ is self-adjoint,

$$\| e - e_n \| = \rho(e - e_n) \leq \sup_{t \in \mathbb{R}} \frac{1}{1 + nt^2} = 1,$$

ρ being the spectral radius. Hence $\| e - e_n \| \leq 1$ $\forall n \in \mathbb{N}$. So

$$\| ze_n - z \|^2 = \| z(e_n - e) \|^2 = \| (e_n - e)z^2(e_n - e) \| \leq \| z^2(e - e_n) \|$$

$$\leq \sup_{t \in \mathbb{R}} \frac{t^2}{1 + nt^2} \leq \frac{1}{n}$$

therefore $\lim \| ze_n - z \| = 0$.

So we have

$$\| xe_n - x \|^2 = \| x(e_n - e) \|^2 \leq \| (e_n - e)x^*x(e_n - e) \| \leq \| z(e_n - e) \| \to 0. \qquad \square$$

Now we may pass to the proof of Theorem 6.2

Proof of theorem 6.2 The ideal $\mathcal{I}$ is self-adjoint: let $x \in \mathcal{I}$ then by lemma 6.3 there is a sequence e_n of elements in $\mathcal{I}$ such that $e_n^* = e_n$ and $\lim e_n x = x$. Taking adjoint we have

$$(e_n x)^* = x^* e_n^* = x^* e_n$$

Since $\mathcal{I}$ is a two sided ideal and $e_n \in \mathcal{I}$, we have $x^* e_n \in \mathcal{I}\, \forall n$. Therefore $\lim x^* e_n = x^*$ and $\mathcal{I}$ is closed $\Longrightarrow x^* \in \mathcal{I}$.

Now fix $x \in \mathcal{U}$ and let $E = \{u \in \mathcal{I} : u^* = u \text{ and } \sigma(u) \subset [0,1]\}$. We will first prove that

$$\| \bar{x} \| = \inf_{t \in \mathcal{I}} \| x - t \| = \inf_{u \in E} \| x - xu \|$$

Clearly $\| \bar{x} \| \leq \inf_{u \in E} \| x - ux \|$. Fix $k \in \mathcal{I}$ then by lemma 6.6 there is a sequence $u_n \in E$ such that $\lim \| k(e - u_n) \| = 0$, and $\| e - u_n \| \leq 1$. Consider

$$\| x + k \| \geq \liminf_n \| (x+k)(e - u_n) \| = \liminf_n \| x(e - u_n) + k(e - u_n) \| =$$

$$\liminf_n \| x(e - u_n) \| .$$

Hence we have $\| \bar{x} \| = \inf_{u \in E} \| x - xu \|$. Now consider

$$\| \bar{x} \|^2 = \inf_{u \in E} \| x - xu \|^2 = \inf_{u \in E} \| x(e-u) \|^2 \leq \inf_{u \in E} \| (e-u) x^* x (e-u) \|^2$$

$$\leq \inf_{u \in E} \| x^* x (e-u) \|^2 = \| \bar{x}^* \bar{x} \| .$$

As a result we have

$$\| \bar{x} \|^2 \leq \| \bar{x}^* \bar{x} \| \qquad \square$$

CHAPTER 7

AN ALGEBRA OF INTEGRAL OPERATORS

Bessel functions G_k are functions defined on $\mathbb{R}$ whose Fourier transforms are given in the following way

$$\hat{G}_k(t) = \frac{1}{(4\pi \mid t \mid^2 + 1)^k}$$

Bessel functions have the following properties:

(i) $\int_{\mathbb{R}} G_k(x)dx = 1 \ k \geq 0$

(ii) $G_k(x) > 0 \ x \in \mathbb{R}$

(iii) $G_k * G_m = G_{k+m}$

The third of these properties is follows from the fact that the Fourier transform is one to one and converts convolution to multiplication. For more information on this class of functions see [22].

Now we associate convolution operators to Bessel functions by defining $H_k f = G_k * f$ that is

$$H_k(f)(x) = G_k * f(x) = \int_{-\infty}^{\infty} G_k(x-y)f(y)dy, \quad x \in \mathbb{R}.$$

Since $\| f * g \|_p \leq \| f \|_1 \| g \|_p$, H_k is bounded on L^p. We note some important properties of the convolution operators H_k

1. Since the G_k's are real valued and positive the H_k's are self-adjoint operators on L^2.

2. Since the linear span $O = \{\lambda_0 I + \sum_{k=1}^{n} \lambda_k H_k : \lambda_k \in \mathbb{C}\}$ contains the constant multiples of identity, the linear span of functions $\{\lambda_0 + \sum_{k=1}^{n} \lambda_k \hat{G}_k : \lambda_k \in \mathbb{C}\}$ separates the points of $[0, \infty]$ and O is closed under taking adjoint, in the view of Stone-Weierstrass theorem O is dense in $\mathcal{C}_p$.

The following proposition and its proof is adapted from [20]. Recall that $\tilde{\mathbb{R}}$ denotes the one point compactification of $\mathbb{R}$.

Proposition 7.1 Let $a \in C(\tilde{\mathbb{R}})$, $T \in \mathcal{B}(L^p(\mathbb{R}))$ be an operator of convolution type i.e for some $k \in L^1(\mathbb{R})$

$$T(f)(x) = k * f(x) = \int_{-\infty}^{\infty} k(x-y) f(y) dy$$

and let M_a be the operator of multiplication by a

$$M_a f(x) = a(x) f(x).$$

Then $M_a.T - T.M_a$ is compact on $L^p(\mathbb{R})$ for all $1 \leq p < \infty$.

Proof Since the linear span of H_k's are dense in $L^1(\mathbb{R})$, it is enough to show that $M_a H_k - H_k M_a$ is compact for all k. Since $a \in C(\tilde{\mathbb{R}})$, for all $\varepsilon > 0$ there is $b \in C(\tilde{\mathbb{R}})$ s.t. $b(x)$ is constant for all x s.t. $\mid x \mid \geq M$ for some M positive and $\mid a(x) - b(x) \mid \leq \varepsilon \qquad \forall x \in \mathbb{R}$ So we have

$$\| (M_a H_k - H_k M_a) - (M_b H_k - H_k M_b) \|_{L^p} \leq 2\varepsilon \| H_k \|_{L^p}$$

So it suffices to prove the compactness of $\mathcal{G} = M_b H_k - H_k M_b$ where $b \in C(\tilde{\mathbb{R}})$ is constant for all x such that $\mid x \mid \geq M$ for some M positive.

$$\mathcal{G}(f)(x) = \int_{-\infty}^{\infty} K(x,y) f(y) dy$$

where $K(x,y) = (b(x) - b(y)) G_k(x-y)$. Since b is constant $\forall x$ s.t. $\mid x \mid > M$,

$K(x,y)=0$ for all x, y s.t. $\min(\mid x\mid, \mid y\mid) > M$. For fixed $A > M$ set

$$K_A(x,y) = \begin{cases} K(x,y) & \text{if} \quad \max\{\mid x\mid, \mid y\mid\} < A \\ 0 \quad \text{otherwise} \end{cases}$$

Since $(\int\int \mid K_A(x,y)\mid^p dxdy)^{\frac{1}{p}} < \infty$, the operator

$$K_A(f)(x) = \int_{-\infty}^{\infty} K_A(x,y)f(y)dy$$

is compact. Now we show that $K_A \longrightarrow \mathcal{G}$ in operator norm as $A \to +\infty$

$$\mathcal{G}(f)(x) - K_A(f)(x) = \int_{-\infty}^{\infty}(K(x,y) - K_A(x,y))f(y)dy$$

$$= \int_{|y|\geq A} K(x,y)f(y)dy + \chi_{\{x:|x|>A\}}\int_{-\infty}^{\infty} K(x,y)f(y)dy$$

Now take $p \neq 1$. By Holder's inequality the first integral is estimated as follows

$$\| \int_{|y|\geq A} K(x,y)f(y)dy \|_p = (\int_{-\infty}^{\infty} \mid \int_{|y|\geq A} K(x,y)f(y)dy \mid^p dx)^{\frac{1}{p}}$$

$$= (\int_{-M}^{M} \mid \int_{|y|\geq A} K(x,y)f(y)dy \mid^p dx)^{\frac{1}{p}}$$

$$\leq \sup_{|x|<M} (\int_{|y|\geq A} \mid K(x,y)\mid^q dy)^{\frac{1}{q}} \| f \|_p$$

For $p = 1$ we have the estimate

$$\| \int_{|y|\geq A} K(x,y)f(y)dy \|_1 \leq \sup_{|x|<M} (\sup_{|y|\geq A} \mid K(x,y)\mid) \| f \|_1$$

Similarly for $p \neq 1$ we have

$$\| \chi_{\{x:|x|>A\}} \int_{-\infty}^{\infty} K(x,y)f(y)dy \|_p \leq \sup_{|y|<M} (\int_{|x|\geq A} \mid K(x,y)\mid^p dx)^{\frac{1}{p}} \| f \|_p$$

For $p = 1$ we have

$$\| \chi_{\{x:|x|>A\}} \int_{-\infty}^{\infty} K(x,y)f(y)dy \|_1 \leq \sup_{|y|<M} (\sup_{|x|\geq A} \mid K(x,y) \mid) \| f \|_1$$

Since b is bounded and G_k's decay at infinity,
$\sup_{|x|<M}(\int_{|y|\geq A} \mid K(x,y) \mid^q dy)^{\frac{1}{q}}$ and $\sup_{|y|<M}(\int_{|x|\geq A} \mid K(x,y) \mid^p dx)^{\frac{1}{p}}$ converge to zero as $A \to +\infty$. Similarly
$\sup_{|x|<M}(\sup_{|y|\geq A} \mid K(x,y) \mid)$ and $\sup_{|y|<M}(\sup_{|x|\geq A} \mid K(x,y) \mid)$ converge to zero as $A \to +\infty$ Hence $K_A \longrightarrow \mathcal{G}$ as $A \to +\infty$ and $\mathcal{G}$ is compact $\square$

The following generalization of Proposition 7.1 is due to H.O.Cordes and is taken from H.O.Cordes, "On Compactness of commutators of Multiplications and Convolutions, and Boundedness of Pseudo-differential Operators" Journal of Functional Analysis 18, p.115-131 (1975):

Theorem(Cordes,1975) Let $a, b \in C(\mathbb{R}^n)$ be bounded functions, let $a(M)f(x) = a(x)f(x)$ be the multiplication operator by a and $b(D) = \mathcal{F}^{-1}b(M)\mathcal{F}$ be the Fourier multiplier by b. Let

$$cm_{x,h}(a) = \sup\{\mid a(x+t) - a(x) \mid : \mid t \mid \leq h\}$$

and $cm_x(a) = cm_{x,1}(a)$ be the continuity modulus at x. If

$$cm_x(a) \to 0, \quad cm_x(b) \to 0 \quad \text{as} \quad \mid x \mid \to \infty$$

then the commutator $[a(M), b(D)] = a(M)b(D) - b(D)a(M)$ is a compact operator on $L^2(\mathbb{R}^n)$.

Let $P : H^p \to H^p$ be the operator by multiplication by $(x-i)/(x+i)$ that is

$$P(f)(x) = (\frac{x-i}{x+i})f(x).$$

So we have $PT - TP$ is compact on H^p for all $T \in \mathcal{C}_p$. Also $P^*T - TP^*$ is compact on H^2 for all $T \in \mathcal{C}_2$ where P^* is the adjoint of P on H^2.

Since P is the shift operator we find out P^* as

$$P^*(f)(x) = (\frac{x+i}{x-i})(f(x) - \frac{2if(i)}{x+i}).$$

Observe that $P^*P = I$ and $PP^*(f)(x) = f(x) - \frac{2if(i)}{x+i}$. Hence $(P^*P - PP^*)(f)(x) = \frac{2if(i)}{x+i}$ is a rank one operator.

Now we are ready to construct our algebra of operators: Let $\mathcal{M}_p$ be the closure of the algebraic linear span generated by the operators I the identity, P and P^* where P and P^* are as above and $\mathcal{C}_p$ be the closure of the algebra generated by the identity and L^1 convolution operators.

We take $\mathcal{A}_p = [\mathcal{C}_p, \mathcal{M}_p, \mathcal{K}_p]$ the subalgebra of $\mathcal{B}(H^p)$ generated by $\mathcal{C}_p$, $\mathcal{M}_p$ and $\mathcal{K}_p$ the space of all compact operators on H^p. Since $\mathcal{K}_p$ is a two-sided closed ideal in $\mathcal{B}(H^p)$, it is a two-sided closed ideal in $\mathcal{A}_p$ as well. Now we will see that $\mathcal{A}_p/\mathcal{K}_p$ is a commutative Banach algebra with identity

Lemma 7.2. Let $\mathcal{A}_p = [\mathcal{C}_p, \mathcal{M}_p, \mathcal{K}_p]$ be the subalgebra of $\mathcal{B}(H^p)$ generated by $\mathcal{C}_p$ the convolution operators, $\mathcal{M}_p$ as above and $\mathcal{K}_p$ the space of all compact operators on H^p as above and let $1 < p < \infty$. Then the quotient algebra $\mathcal{A}_p/\mathcal{K}_p$ is commutative.

Proof. Let $S \in \mathcal{A}_p$ such that

$$S = \sum_{n=1}^{N} \lambda_n (P^*)^n B_n + \alpha_1 P^* + \lambda_0 I + \alpha_2 P + \sum_{n=0}^{M} \mu_n P^n D_n + K.$$

where B_n, $D_n \in \mathcal{C}_p$, $K \in \mathcal{K}_p$ and let $[S]$ denote the coset of S in $\mathcal{A}_p/\mathcal{K}_p$. Let $S' \in \mathcal{A}_p$ such that

$$S' = \sum_{n=1}^{N} \lambda_n B_n (P^*)^n + \alpha_1 P^* + \lambda_0 I + \alpha_2 P + \sum_{n=0}^{M} \mu_n D_n P^n + K.$$

Then by Proposition 7.1. we immediately see that $(S - S') \in \mathcal{K}_p$. Hence $[S] = [S']$.

So without loss of generality it is enough to show that any two operators S_1 and $S_2 \in \mathcal{A}_p$ of the form

$$S_1 = \sum_{n=1}^{N} \lambda_n (P^*)^n B_n + \alpha_1 P^* + \lambda_0 I + \alpha_2 P + \sum_{n=0}^{M} \mu_n P^n D_n + K_1$$

and

$$S_2 = \sum_{n=1}^{N} \lambda'_n (P^*)^n B_n + \alpha'_1 P^* + \lambda'_0 I + \alpha'_2 P \sum_{n=0}^{M} \mu'_n P^n D_n + K_2$$

commute. We again see that $S_1S_2 - S_2S_1$ consists of commutator terms which are compact by Proposition 7.1.,and by the fact that $PP^* - P^*P$ is an operator of rank 1. Hence we have $(S_1S_2 - S_2S_1) \in \mathcal{K}_p$ and $[S_1S_2] = [S_2S_1]$. □

In view of theorem 6.2 for $p = 2$, $\mathcal{A}_2/\mathcal{K}_2$ is a C* algebra under the quotient norm and by Lemma 7.2., $\mathcal{A}_2/\mathcal{K}_2$ is a commutative C* algebra so is isometrically isomorphic to $C(X)$ where X is the maximal ideal space of $\mathcal{A}_p/\mathcal{K}_p$. The quotient algebra $\mathcal{A}_2/\mathcal{K}_2$ is an algebra with symbol in the sense Definition 6.1. This can be easily seen as follows.

(i) The subalgebra $\mathcal{C}_2$ of $\mathcal{B}(H^2)$ is a commutative * subalgebra,

(ii) $\mathcal{C}_2 \cap K(H^2) = \{0\}$ since for any $T \in \mathcal{C}_2$ there is a continuous function $a \in C([0,\infty])$ such that $T = \mathcal{F}^{-1} M_a \mathcal{F}$ and the multiplication operator M_a is never compact unless a is identically 0,(Recall that $\mathcal{F}$ denotes the Fourier transform)

(iii) $P_1P_2 - P_2P_1 \in K(H^2)$, $\quad P_1,P_2 \in \mathcal{M}_2$,

(iv) and $PT - TP \in K(H^2)$, $\quad P \in \mathcal{M}_2$, $T \in \mathcal{C}_2$.

At this point it is natural to ask how to characterize X the maximal ideal space, and the Gelfand transform Γ of $\mathcal{A}_p/\mathcal{K}_p$. In the following lemma we characterize the maximal ideal space X of $\mathcal{A}_p/\mathcal{K}_p$.

Lemma 7.3. Let X be the maximal ideal space of $\mathcal{A}_p/\mathcal{K}_p$. Then $X \cong \tilde{\mathbb{R}} \times [0,\infty]$.

Proof. Take $\Lambda \in X$ and examine how it acts on $\mathcal{A}_p/\mathcal{K}_p$. Since $\mathcal{A}_p$ is generated by the subalgebras $\mathcal{M}_p$ and, $\mathcal{C}_p$ Λ is completely determined by its values on the cosets of the elements of these algebras.

The values of Λ on the cosets of elements in $\mathcal{M}_p$ are completely determined by $\Lambda([P])$ and $\Lambda([P^*])$. Since $I - P^*P$ is compact we have

$$\Lambda([P^*P]) = \Lambda([P^*])\Lambda([P]) = 1.$$

We also have

$$\| P^* \|_e = \| P \|_e = 1$$

and

$$\Lambda([P^*]) \leq \| P^* \|, \Lambda([P]) \leq \| P \|_e .$$

These together give

$$\Lambda([P^*]) = \left(\frac{x_0 + i}{x_0 - i}\right), \Lambda([P]) = \left(\frac{x_0 - i}{x_0 + i}\right)$$

for some $x_0 \in \tilde{\mathbb{R}}$.

The values of Λ on the cosets of the elements in $\mathcal{C}_p$ are determined by its values on $[H_\alpha]$'s. The action of Λ on $[H_\alpha]$'s is given by Fourier transform since these are convolution operators and Fourier transform is the only norm decreasing transform that converts convolution to multiplication i.e.

$$\Lambda([H_\alpha]) = \widehat{G_\alpha}(t_0)$$

for some $t_0 \in [0, \infty]$ by Paley-Wiener theorem.

Therefore for any $\Lambda \in X$ there is a unique $(x_0, t_0) \in \tilde{\mathbb{R}} \times [0, \infty]$ such that for all $S \in \mathcal{A}_p$ given as

$$S = \sum_{n=1}^{N} \lambda_n (P^*)^n H_{\alpha_n} + \eta_1 P^* + \lambda_0 I + \eta_2 P + \sum_{n=0}^{M} \mu_n P^n H_{\beta_n} + K$$

we have

$$\Lambda([S]) = \sum_{n=1}^{N} \lambda_n \left(\frac{x_0+i}{x_0-i}\right)^n \widehat{G_{\alpha_n}}(t_0) + \eta_1 \left(\frac{x_0+i}{x_0-i}\right)$$

$$+\lambda_0 + \eta_2 \left(\frac{x_0-i}{x_0+i}\right) + \sum_{n=0}^{M} \mu_n \left(\frac{x_0-i}{x_0+i}\right)^n \widehat{G_{\beta_n}}(t_0).$$

Hence we have $X \cong \tilde{\mathbb{R}} \times [0, \infty]$. $\square$

In the following theorem we characterize the Gelfand transform of $\mathcal{A}_p/\mathcal{K}_p$

Theorem 7.4. Let $\Gamma : \mathcal{A}_p/\mathcal{K}_p \to C(\tilde{\mathbb{R}} \times [0, \infty])$ be the Gelfand transform of $\mathcal{A}_p/\mathcal{K}_p$. Then for $S \in \mathcal{A}_p$ in the following form

$$S = \sum_{n=1}^{N} \lambda_n (P^*)^n H_{\alpha_n} + \eta_1 P^* + \lambda_0 I + \eta_2 P + \sum_{n=0}^{M} \mu_n P^n H_{\beta_n} + K$$

the Gelfand transform $\Gamma[S]$ of S is in $C(\tilde{\mathbb{R}} \times [0, \infty])$ and is given by the formula

$$(\Gamma[S])(x,t) = \sum_{n=1}^{N} \lambda_n \left(\frac{x+i}{x-i}\right)^n \widehat{G_{\alpha_n}}(t) + \eta_1 \left(\frac{x+i}{x-i}\right)$$

$$+\lambda_0 I + \eta_2 \left(\frac{x-i}{x+i}\right) + \sum_{n=0}^{M} \mu_n \left(\frac{x-i}{x+i}\right)^n \widehat{G_{\beta_n}}(t).$$

Proof. The proof easily follows by Lemma 3.1. and the definition of the Gelfand transform. $\square$.

We close this section by a theorem that we will use to characterize essential spectra of our composition operators

Theorem 7.5. Let $T \in \mathcal{A}_p$, $1 \leq p < \infty$, then for $1 < p < \infty$

$$\sigma_e(T) \subseteq \overline{\Gamma[T](\tilde{\mathbb{R}} \times [0, \infty])}.$$

Moreover for $p = 2$ we have

$$\sigma_e(T) = \overline{\Gamma[T](\tilde{\mathbb{R}} \times [0, \infty])}$$

Proof. Since $\mathcal{A}_p/\mathcal{K}_p$ is a commutative Banach algebra with identity we have

$$\sigma_{\mathcal{A}_p/\mathcal{K}_p}([T]) = \overline{\Gamma[T](\tilde{\mathbb{R}} \times [0, \infty])}.$$

And $\mathcal{A}_p/\mathcal{K}_p$ is a subalgebra of $\mathcal{B}(H^p)/K(H^p)$ we have

$$\sigma_e(T) = \sigma_{\mathcal{B}(H^p)/K(H^p)}([T]) \subseteq \sigma_{\mathcal{A}_p/\mathcal{K}_p}([T]) = \overline{\Gamma[T](\tilde{\mathbb{R}} \times [0, \infty])}.$$

For $p = 2$ $\mathcal{A}_2/\mathcal{K}_2$ is a C* algebra and a * subalgebra of the C* algebra $\mathcal{B}(H^2)/K(H^2)$. Because of that we have

$$\sigma_{\mathcal{B}(H^2)/K(H^2)}([T]) = \sigma_{\mathcal{A}_2/\mathcal{K}_2}([T]).$$

Hence we have for $p = 2$

$$\sigma_e(T) = \overline{\Gamma[T](\tilde{\mathbb{R}} \times [0, \infty])}. \qquad \square$$

CHAPTER 8

ESSENTIAL SPECTRA OF COMPOSITION OPERATORS ON HARDY SPACES OF THE UPPER HALF-PLANE

In this chapter we will characterize the essential spectrum of the composition operators $C_\varphi : H^2(\mathbb{H}) \longrightarrow H^2(\mathbb{H})$ that are induced by φ, analytic self-map of the upper half plane, satisfying the following conditions:

a-) The function φ extends analytically across $\mathbb{R}$.

b-) The function $\psi(z) = \varphi(z) - z$ is a bounded analytic function on $\mathbb{H}$ and $\overline{\psi(\mathbb{H})} \subset\subset \mathbb{H}$ i.e. $\overline{\psi(\mathbb{H})}$ has compact closure in $\mathbb{H}$

c-) The function ψ is analytic at ∞

Our strategy in finding essential spectra is to show that if φ satisfies the conditions (a),(b) and (c) above then $C_\varphi \in \mathcal{A}_2$. Since $\mathcal{A}_2/\mathcal{K}_2$ is a commutative C* algebra and a closed subalgebra of $\mathcal{B}(H^2)/K(H^2)$, the essential spectrum of C_φ coincides with the range of the Gelfand transform of C_φ on $\mathcal{A}_2$. For $p \neq 2$ the essential spectrum falls inside the range of the Gelfand transform. Finally we calculate the Gelfand transform of C_φ on $\mathcal{A}_p/\mathcal{K}_p$ using the integral representation of the operator. To show that C_φ is bounded on H^2 and $C_\varphi \in \mathcal{A}_2$ we will first prove the following lemmata, Lemma 8.1 and Lemma 8.2:

Lemma 8.1 Let $p : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be an integrable function that satisfies

$$\int_{-\infty}^{\infty} \sup_{x \in \mathbb{R}} \mid p(x,t) \mid dt < \infty$$

then the following operator $P : L^p(\mathbb{R}) \to L^p(\mathbb{R})$,

$$P(f)(x) = \int_{-\infty}^{\infty} p(x,t) f(x-t) dt$$

is bounded on $L^p(\mathbb{R})$. Moreover the following estimate holds for the L^p operator norm $\| P \|$ of P;

$$\| P \| \leq \int_{-\infty}^{\infty} \sup_{x \in \mathbb{R}} | p(x,t) | \, dt.$$

Proof Set

$$\tilde{p}(t) = \sup_{x \in \mathbb{R}} | p(x,t) | \, .$$

then we have

$$| P(f)(x) | \leq \int_{-\infty}^{\infty} | f(x-t) || p(x,t) | \, dt \leq \int_{-\infty}^{\infty} | f(x-t) | \tilde{p}(t) dt.$$

So we have the following estimate

$$\int_{-\infty}^{\infty} | P(f)(x) |^p \, dx \leq \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} | f(x-t) |^p \, dx)(\tilde{p}(t))^p dt \leq \| f \|_p^p \| \tilde{p} \|_1^p \, .$$

The first inequality follows immediately and the second follows since $\tilde{p}$ is positive. Therefore we have

$$\| P(f) \|_p \leq \| \tilde{p} \|_1 \| f \|_p \, . \qquad \square$$

Lemma 8.2 Let $\psi : \mathbb{H} \to \mathbb{H}$ be an analytic function such that $\overline{\psi(\mathbb{H})} \subset\subset \mathbb{H}$ i.e. $\overline{\psi(\mathbb{H})}$ is a compact subset of $\mathbb{H}$. Then for the function f defined as follows

$$f(x,t) = \frac{\psi(x) - \psi(i)}{(\psi(x) + t)(\psi(i) + t)}$$

we have for any small $\varepsilon > 0$ there is some $N > 0$ such that

$$\int_{\mathbb{R} - [-N,N]} \sup_{x \in \mathbb{R}} | f(x,t) | \, dt < \varepsilon$$

Proof Let $K = \overline{\psi(\mathbb{H})}$, K is a compact subset of $\mathbb{H}$. Let $\kappa = \inf_{z \in K} \Im(z)$ and

$\Gamma = \{z : \Im(z) = \kappa\} \cap K$, since K is compact Γ is nonempty. Set $\beta = \inf_{z\in\Gamma} \Re(z)$, $\gamma = \sup_{z\in\Gamma} \Re(z)$, and $\delta = (\beta + \gamma)/2$. Then we have

$$\int_{-\infty}^{\infty} \frac{dt}{\mid t + \psi(i) \mid\mid t + \psi(x) \mid} = \int_{-\infty}^{\infty} \frac{dt}{\mid t + \psi(i) - \delta \mid\mid t + \psi(x) - \delta \mid}$$

by translation invariance of the Lebesgue measure. Set $\tilde{K} = -\delta + K$. Let $\varepsilon > 0$ such that $\kappa - \varepsilon > \kappa/2$. Then there is a disc $\mathbb{D}(i\alpha, \alpha - \kappa + \varepsilon)$ centered at $i\alpha$ with radius $\alpha - \kappa + \varepsilon$ such that $\tilde{K} \subset\subset \mathbb{D}(i\alpha, \alpha - \kappa + \varepsilon)$ i.e. $\tilde{K}$ is compactly contained in $\mathbb{D}(i\alpha, \alpha - \kappa + \varepsilon)$. Let $2\alpha/\kappa = M > 1$ then

$$\frac{t + i\alpha}{\frac{t}{M} + i\frac{\kappa}{2}} = M > 1$$

and so we have

$$\inf_{z\in\tilde{K}} \mid z + t \mid > \inf_{z\in\mathbb{D}(i\alpha,\alpha-\kappa+\varepsilon)} \mid z + t \mid > \mid \frac{t}{M} + i\frac{\kappa}{2} \mid$$

and this implies

$$\frac{1}{\mid \psi(i) + t - \delta \mid\mid \psi(x) + t - \delta \mid} < \frac{M}{\mid t + i\frac{\kappa M}{2} \mid^2}$$

for all $x \in \mathbb{R}$. And there is $S > 0$ such that

$$\mid \psi(x) - \psi(i) \mid < S$$

for all $x \in \mathbb{R}$. Combining these we have

$$\sup_{x\in\mathbb{R}} \frac{\mid \psi(x) - \psi(i) \mid}{\mid \psi(i) + t - \delta \mid\mid \psi(x) + t - \delta \mid} < \frac{MS}{\mid t + i\frac{\kappa M}{2} \mid^2}.$$

Now fix small $\varepsilon > 0$, then there exists a $K > \delta$ such that

$$\int_{\mathbb{R}-[-K,K]} \frac{MSdt}{\mid t + i\frac{\kappa M}{2} \mid^2} < \varepsilon.$$

Taking $N = K + \delta$ we have

$$\int_{\mathbb{R}-[-N,N]} \sup_{x\in\mathbb{R}} \frac{|\ \psi(x)-\psi(i)\ |}{|\ \psi(i)+t\ ||\ \psi(x)+t\ |}dt \leq$$

$$\int_{\mathbb{R}-[-K,K]} \sup_{x\in\mathbb{R}} \frac{|\ (\psi(x)-\psi(i))\ |}{|\ \psi(i)+t-\delta\ ||\ \psi(x)+t-\delta\ |}dt$$

$$\leq \int_{\mathbb{R}-[-K,K]} \frac{MSdt}{|\ t+i\frac{\kappa M}{2}\ |^2} < \varepsilon.$$

As a result we have for all $\varepsilon > 0$ there is $N > 0$ such that

$$\int_{\mathbb{R}-[-N,N]} \sup_{x\in\mathbb{R}} |\ f(x,t)\ |\ dt < \varepsilon. \qquad \square$$

Lemmas 8.1 and 8.2 can be used to show the boundedness of C_φ where φ satisfies the conditions (a),(b) and (c) in the following way: Consider $C_\varphi - T_{\psi(i)}$ where $T_\alpha(f)(x) = f(x+\alpha)$;

$$(C_\varphi - T_{\psi(i)})(f)(x) = \int_{-\infty}^{\infty} \frac{f(x-t)dt}{-\psi(x)-t} - \int_{-\infty}^{\infty} \frac{f(x-t)dt}{-\psi(i)-t}$$

$$= \int_{-\infty}^{\infty} \frac{\psi(x)-\psi(i)}{(\psi(x)+t)(\psi(i)+t)} f(x-t)dt.$$

By lemma 8.3

$$\int_{-\infty}^{\infty} \sup_{x\in\mathbb{R}} \frac{|\ \psi(x)-\psi(i)\ |}{|\ (\psi(x)+t)(\psi(i)+t)\ |}dt < \infty.$$

Hence lemma 8.2 implies that $C_\varphi - T_{\psi(i)}$ is a bounded operator on L^p and hence on H^p. Since $T_{\psi(i)}$ is bounded, C_φ is also bounded.

Now we will use lemma 8.1 and lemma 8.2 to prove the following theorem:

Theorem 8.3 Let $1 < p < \infty$ and let $C_\varphi : H^p(\mathbb{H}) \to H^p(\mathbb{H})$ be the composition operator induced by the analytic function $\varphi : \mathbb{H} \to \mathbb{H}$ satisfying

1. φ is analytic across $\mathbb{R}$
2. For the function $\psi(z) = \varphi(z) - z$ we have $\overline{\psi(\mathbb{H})} \subset\subset \mathbb{H}$
3. The function ψ is analytic on $\tilde{\mathbb{R}}$.

Then $C_\varphi \in \mathcal{A}_2$.

Proof. Let $k : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be

$$k(x,t) = \frac{1}{2\pi i} \frac{1}{x - \varphi(x) - t}.$$

By Theorem 3.3, we write C_φ as a singular integral operator with variable kernel

$$C_\varphi(f)(x) = \int_{-\infty}^{\infty} k(x, x-y) f(y) dy.$$

Since $\psi(z) = \varphi(z) - z$ is bounded and analytic at infinity, $k(., t)$ is also analytic and keeping t fixed $k(., t)$ has a unique power series expansion in powers of $\frac{x-i}{x+i}$ in the following way:

$$k(z,t) = \sum_{n=0}^{\infty} b_n(t) (\frac{z-i}{z+i})^n \qquad \forall t \in \mathbb{R}$$

where $b_n(t)$'s are continuous functions depending on the partial derivatives of $k(z,t)$ with respect to z at $z = i$. Since φ extends analytically across $\mathbb{R}$ the following Taylor series expansion is uniform in $x \in \mathbb{R}$ pointwisely in t:

$$\frac{1}{x - \varphi(x) - t} = \sum_{n=0}^{\infty} (\frac{x-i}{x+i})^n b_n(t)$$

Using this power series expansion we will show that $\forall\ \epsilon > 0$ there exists an integer N and $\tilde{b_n} \in L^1$ such that the partial sum

$$k_N(x,t) = \sum_{n=0}^{N} (\frac{x-i}{x+i})^n \tilde{b_n}(t)$$

satisfies

$$\| \sup_{x \in \mathbb{R}} | \ k(x,\cdot) - k_N(x,\cdot) \ | \quad \|_1 < \epsilon.$$

And this will imply that $C_\varphi \in \mathcal{A}_p$ by lemma 8.1. Since the kernel $\frac{1}{x-\varphi(x)-t}$ is not in $L^1(dt)$ in t variable, we take the constant term in the Taylor series expansion to the other side and consider the following

$$\frac{1}{\psi(i)+t}-\frac{1}{\psi(x)+t}=\frac{\psi(x)-\psi(i)}{(\psi(x)+t)(\psi(i)+t)}=\sum_{n=1}^{\infty}(\frac{x-i}{x+i})^n b_n(t)$$

$\forall\, t$ uniformly in x. Here $\psi(x)=\varphi(x)-x$. The function $\frac{\psi(x)-\psi(i)}{(\psi(x)+t)(\psi(i)+t)}$ is in $L^1(dt)$ in t variable and moreover by Lemma 8.2 we have $\forall\, \varepsilon>0\ \exists\, M>0$ such that

$$\int_{-\infty}^{\infty}\sup_{x\in\mathbb{R}}\frac{\mid \psi(x)-\psi(i)\mid dt}{\mid(\psi(x)+t)(\psi(i)+t)\mid}-\int_{-M}^{M}\sup_{x\in\mathbb{R}}\frac{\mid \psi(x)-\psi(i)\mid dt}{\mid(\psi(x)+t)(\psi(i)+t)\mid}<\frac{\varepsilon}{2}.$$

On the other hand since $b_n(t)$'s are continuous we have $\forall\, \epsilon'>0\ \exists\, \delta>0$ and $\exists$ $N>0$ s.t.

$$\mid t-t_0\mid<\delta\Rightarrow\mid\frac{\psi(x)-\psi(i)}{(\psi(x)+t)(\psi(i)+t)}-\sum_{n=1}^{N}(\frac{x-i}{x+i})^n b_n(t)\mid<\varepsilon'$$

$\forall\, x\in\mathbb{R}$. Since $[-M,M]$ is compact one also has $\forall\, \varepsilon'>0\ \exists\, N\in\mathbb{N}$ s.t. $\forall\, t\in[-M,M]$

$$\sup_{x\in\mathbb{R}}\mid\frac{\psi(x)-\psi(i)}{(\psi(x)+t)(\psi(i)+t)}-\sum_{n=1}^{N}(\frac{x-i}{x+i})^n b_n(t)\mid<\varepsilon'.$$

Take $\varepsilon'=\frac{\varepsilon}{4M}$. Then we have

$$\int_{-M}^{M}\sup_{x\in\mathbb{R}}\mid\frac{\psi(x)-\psi(i)}{(\psi(x)+t)(\psi(i)+t)}-\sum_{n=1}^{N}(\frac{x-i}{x+i})^n b_n(t)\mid dt<\frac{\varepsilon}{2}.$$

Now take $\tilde{b_n}=\chi_{[-M,M]}.b_n$. Therefore we have $\forall\, \varepsilon>0\ \exists\, N\in\mathbb{N}$ and $\tilde{b_n}\in L^1$ such that

$$\int_{-\infty}^{\infty}\sup_{x\in\mathbb{R}}\mid\frac{\psi(x)-\psi(i)}{(\psi(x)+t)(\psi(i)+t)}-\sum_{n=1}^{N}(\frac{x-i}{x+i})^n \tilde{b_n}(t)\mid dt<\varepsilon.$$

Here $\chi_{[-M,M]}$ denotes the characteristic function of $[-M,M]$. So setting $p(x,t)=\frac{\psi(x)-\psi(i)}{(\psi(x)+t)(\psi(i)+t)}$ we have

$$\|\sup_{x\in\mathbb{R}}\mid p(x,\cdot)-\sum_{n=1}^{N}(\frac{x-i}{x+i})^n\tilde{b_n}(\cdot)\mid\quad\|_1<\varepsilon.$$

So we have $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ and $\tilde{b_n} \in L^1$ such that

$$\| \sup_{x\in\mathbb{R}} \mid k(x,\cdot) - \sum_{n=0}^{N} (\frac{x-i}{x+i})^n \tilde{b_n}(\cdot) \mid \quad \|_1 < \varepsilon$$

where $k(x,t) = \frac{1}{x-\varphi(x)-t}$ and $\tilde{b_0}(t) = b_0(t) = \frac{1}{i-\varphi(i)-t}$. Therefore $C_\varphi - T_{\psi(i)} \in \mathcal{A}_p$ where $T_{\psi(i)}f(z) = f(z+\psi(i))$. And since $T_{\psi(i)} \in \mathcal{A}_2$ we have $C_\varphi \in \mathcal{A}_2$ $\square$

And the Gelfand transform of C_φ is computed as

$$\Gamma(C_\varphi) = \hat{k}(x,\cdot) = e^{i(\varphi(x)-x)t}$$

For $p = 2$ since $\mathcal{A}_2/K(H^2(\mathbb{H}))$ is a commutative C* algebra with identity the essential spectrum $\sigma_e(C_\varphi)$ of C_φ coincides with the range of the Gelfand transform of C_φ. Hence we have

$$\sigma_e(C_\varphi\mid_{H^2}) = \overline{\{e^{i(\varphi(x)-x)t} : x \in \mathbb{R}, t \in [0,\infty)\}}.$$

Since $\mathcal{A}_2/\mathcal{K}_2$ is commutative for any $T \in \mathcal{A}_2$, $TT^* - T^*T$ is compact i.e. any $T \in \mathcal{A}_2$ is essentially normal. Hence for φ satisfying the conditions (a), (b) and (c) above $C_\varphi : H^2 \to H^2$ is essentially normal.

So we summarize our result as the following theorem :

Theorem A. Let $\varphi : \mathbb{H} \to \mathbb{H}$ be analytic and extend analytically across $\mathbb{R}$. Let φ also satisfy the following:

(a) The function $\psi(z) = \varphi(z) - z$ is a bounded analytic function on $\mathbb{H}$ that is analytic also at ∞,

(b) the imaginary part of ψ satisfies $\Im(\psi(z)) \geq M > 0$ for all $z \in \mathbb{H}$ for some $M > 0$.

Then for $1 < p < \infty$,

i-) $C_\varphi : H^p(\mathbb{H}) \to H^p(\mathbb{H})$ is bounded and $C_\varphi : H^2 \to H^2$ is essentially normal.

ii-) The essential spectrum of $C_\varphi : H^2(\mathbb{H}) \to H^2(\mathbb{H})$ is given by

$$\sigma_e(C_\varphi) = \overline{\{e^{i\psi(x)t} : x \in \mathbb{R}, t \in (0, \infty)\}}.$$

CHAPTER 9

ESSENTIAL SPECTRA OF COMPOSITION OPERATORS ON HARDY SPACES OF THE UNIT DISC

For the unit disc under the conditions on ψ, $\frac{\psi(x)+i}{x+i}$ is uniformly bounded in x where $\psi = \tau^{-1}\varphi\tau$, τ being the Cayley transform.And we have $\lim_{x\to\pm\infty} \frac{\psi(x)+i}{x+i} = 1$ for $x \in \mathbb{R}$. Hence we have $\frac{\psi(x)+i}{x+i} \in C(\tilde{\mathbb{R}})$. Now let $T \in \mathcal{B}(H^p)$ such that $T = ((\psi(x)+i)/(x+i))^{2/p} C_\psi$. Then as we had seen in section 2, the operator C_φ on the Hardy space of the unit disc is unitarily equivalent to T. So we have

$$\sigma_e(C_\varphi) = \sigma_e(T).$$

And since $\frac{\psi(x)+i}{x+i} \in C(\tilde{\mathbb{R}})$, $T \in \mathcal{A}_2$ whenever ψ is analytic across $\mathbb{R}$, the function $\eta(z) = \psi(z) - z$ is a bounded analytic function on $\mathbb{H}$ with $\overline{\eta(\mathbb{H})} \subset\subset \mathbb{H}$, and η is analytic at infinity. So for $p = 2$ we have

$$\sigma_e(C_\varphi) = \overline{\{\left(\frac{\psi(x)+i}{x+i}\right) e^{i(\psi(x)-x)t} : x \in \mathbb{R}, t \in (0,\infty)\}}.$$

Let τ be the Cayley transform,the map that takes the upper half-plane conformally onto the unit disc in a one-to one manner. We observe that if for $\varphi : \mathbb{D} \to \mathbb{D}$ satisfies

$$\psi = \tau^{-1}\varphi\tau, \qquad \psi(z) = z + \eta(z)$$

where $\eta : \mathbb{H} \to \mathbb{H}$ is a bounded analytic function then φ has the following form:

$$\varphi(w) = \frac{2iw + \eta(\frac{i(1-w)}{1+w})(1-w)}{2i + \eta(\frac{i(1-w)}{1+w})(1-w)}$$

So we formulate our result as the following theorem:

Theorem B. If $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic function of the following form

$$\varphi(w) = \frac{w + \eta(w)(1-w)}{1 + \eta(w)(1-w)}$$

where $\eta : \mathbb{D} \to \mathbb{C}$ is a bounded analytic function with $\Re(\eta(w)) > M > 0$ for all $w \in \mathbb{D}$ and η extends analytically across $\mathbb{T}$ then

1-) the operator $C_\varphi : H^2 \to H^2$ is essentially normal,

2-) the essential spectrum $\sigma_e(C_\varphi)$ on $H^2(\mathbb{D})$ is given by

$$\sigma_e(C_\varphi) =$$

$$\overline{\{\left(\frac{x + 2i\eta(e^{i\theta}) + i}{x+i}\right)e^{-2\eta(e^{i\theta})t} : x \in \mathbb{R}, t \in [0,\infty)\}}$$

where

$$e^{i\theta} = (x-i)/(x+i).$$

CHAPTER 10

ESSENTIAL SPECTRA OF COMPOSITION OPERATORS ON SPACES OF BOUNDED ANALYTIC FUNCTIONS

In this final chapter of our thesis we will characterize the essential spectra of composition operators $C_\varphi : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$ where $\varphi(z) = z + b(z)$, the function b is bounded analytic with $b(\mathbb{H}) \subset\subset \mathbb{H}$ and $\lim_{z\to\infty} b(z) = b_0$ exists.

We begin with a special case where the function b is constant i.e. $b(z) \equiv b_0$ $\forall z \in \mathbb{H}$. For determining the spectra of these operators we use semigroup of operators techniques. We will use the following Theorem 10.1 cited from [13,pp.93]:

Let $D \subseteq \mathbb{C}$ be a domain in the complex plane and let X,Y be Banach spaces. Let $\Gamma_2 \subset Y^*$ be a determining manifold for Y i.e. if $y^*(y) = 0$ $\forall y^* \in Y^*$ then $y = 0$. Then $U : D \to \mathcal{B}(X, Y)$ is called holomorphic if the function $f(w) = y^*(U_w x)$ is holomorphic on D $\forall x \in X$, $y^* \in \Gamma_2$.

Theorem 10.1 Let $D \subset \mathbb{C}$ be a domain and $U : D \to \mathcal{B}(X, Y)$ be a function.

If U is holomorphic on D then U is continuous and differentiable on D in the uniform operator topology of $\mathcal{B}(X, Y)$ i.e. for any $z_0 \in D$ and $\varepsilon > 0$ there exists $T \in \mathcal{B}(X, Y)$ and $\delta > 0$ such that $\forall w \in D$ with $\mid w - z_0 \mid< \delta$ we have

$$\| \frac{1}{w - z_0}(U(w) - U(z_0)) - T \|_{\mathcal{B}(X,Y)} < \varepsilon.$$

As a corollary we have if U is holomorphic then $\forall F \in \mathcal{B}(X,Y)$ the function $g(w) = F(U_w)$ is holomorphic.

For $w \in \mathbb{H}$ let $T_w : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$, $T_w f(z) = f(z+w)$. We consider the following algebra of operators

$$\mathcal{B} = \overline{< \{T_w : w \in \mathbb{H}\} \cup \{I\} >}$$

the closure of the linear span of $\{T_w : w \in \mathbb{H}\} \cup \{I\}$ in the operator norm of $\mathcal{B}(H^\infty)$. The algebra $\mathcal{B}$ is a commutative Banach algebra with identity. Let $\mathcal{M}$ be the maximal ideal space of $\mathcal{B}$. Then we have

$$\sigma(T_w) \subseteq \{\Lambda(T_w) : \Lambda \in \mathcal{M}\}$$

by Gelfand theory of commutative Banach algebras.

Fix $\Lambda \in \mathcal{M}$ and consider $g(w) = \Lambda(T_w)$. We will see that g is holomorphic: for that we will use the Theorem 10.1. To apply the Theorem 10.1 we will take $U_w = T_w$, $X = Y = H^\infty(\mathbb{H})$, $\Gamma_1 = \Gamma_2 = \{\delta_z : z \in \mathbb{H}\}$ where $\delta_z(f) = f(z)$. By Hahn-Banach theorem there exists $\tilde{\Lambda} \in \mathcal{B}(H^\infty)^*$ such that $\tilde{\Lambda} \mid_{\mathcal{B}} = \Lambda$. So by Theorem 10.1 if for any $z \in \mathbb{H}$ and $f \in H^\infty(\mathbb{H})$ the function $h(w) = \delta_z(T_w f)$ is holomorphic then the function $g(w) = \tilde{\Lambda}(T_w) = \Lambda(T_w)$ is holomorphic. It is easy to see that h is holomorphic. Hence g is holomorphic. The function g also satisfies the following

$$g(w_1 + w_2) = g(w_1)g(w_2) \qquad \forall w_1, w_2 \in \mathbb{H}$$

and

$$\mid g(w) \mid \leq \parallel T_w \parallel = 1 \qquad \forall w \in \mathbb{H}$$

So we deduce that $g(w) = e^{it_0 w}$ for some $t_0 \in [0, \infty)$. So we have for any $w \in \mathbb{H}$

$$\sigma_e(T_w) \subseteq \sigma(T_w) \subseteq \{e^{itw} : t \in [0, \infty)\} \cup \{0\}$$

Now take $\lambda = e^{it_0 w}$ for some $t_0 \in (0, \infty)$. Then the function $f(z) = e^{it_0 z}$ is in

$H^\infty(\mathbb{H})$ and satisfies

$$T_w f(z) = e^{it_0 w} f(z)$$

So we have $f \in \ker(e^{it_0 w} I - T_w)$ and hence $\lambda = e^{it_0 w} \in \sigma(T_w)$. Therefore

$$\sigma(T_w) = \{e^{itw} : t \in [0, \infty)\} \cup \{0\}$$

Now let $D_w = \{e^{2\pi i \frac{z}{w}} : z \in \mathbb{H}\}$, D_w is the image of $\mathbb{H}$ under a holomorphic map and hence is open with nonempty interior. Consider the following subspace K of H^∞:

$$K = \{f(z) = e^{it_0 z} k(e^{2\pi i \frac{z}{w}}) : k \in H^\infty(D_w)\}$$

Observe that $K \subseteq \ker(e^{it_0 w} I - T_w)$. Hence $\ker(e^{it_0 w} I - T_w)$ is infinite dimensional. This implies that $e^{it_0 w} I - T_w$ is not Fredholm and by Atkinson's theorem we have $e^{it_0 w} \in \sigma_e(T_w)$. So we have

$$\sigma_e(T_w) = \sigma(T_w) = \{e^{itw} : t \in [0, \infty)\} \cup \{0\}$$

Let X be a Banach space and $K(X)$ be the space of all compact operators on X. Take $K \in K(X)$. Since for any $T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$, $\lambda I - T = \lambda I - T - K$ in $\mathcal{B}(X)/K(X)$ we have

$$\sigma_e(T + K) = \sigma_e(T)$$

i.e. the essential spectrum is invariant under compact perturbations. Using this fact and the following Theorem 10.2 we conclude that if $\varphi(z) = z + b(z)$, the function b is bounded analytic with $b(\mathbb{H}) \subset\subset \mathbb{H}$ and $\lim_{z\to\infty} b(z) = b_0$ then for $C_\varphi : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$ we have

$$\sigma_e(C_\varphi) = \sigma_e(T_{b_0}) = \{e^{itb_0} : t \in [0, \infty)\} \cup \{0\}$$

Theorem 10.2 Let $b : \mathbb{H} \to \mathbb{H}$ be a bounded analytic function such that $b(\mathbb{H}) \subset\subset \mathbb{H}$ and let $\lim_{z\to\infty} b(z) = b_0 \in \mathbb{H}$ exists. Let $\varphi(z) = z + b(z)$ and $T_{b_0} f(z) = f(z + b_0)$, $T_{b_0} : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$. Then $C_\varphi - T_{b_0}$ is compact on $H^\infty(\mathbb{H})$.

Proof Take $\{f_n\}_{n=1}^{\infty} \subset H^{\infty}(\mathbb{H})$ such that $\| f_n \|_{\infty} \leq 1$. Consider $K_j = \{x+iy \in \mathbb{H} : | x | \leq j, \frac{1}{j} \leq | y | \leq j\}$, K_j's are compact, $K_{j+1} \supset K_j$ and $\bigcup_{j=1}^{\infty} K_j = \mathbb{H}$. Since $\{f_n\}$ is equibounded and equicontinuous on K_1, K_1 is compact by Arzela Ascoli theorem $\{f_n\}$ has a subsequence $\{f_{n_j}\}$ that converges uniformly on K_1. Applying the same process on K_2 and $\{f_{n_j}\}$, going on iteratively we arrive at a subsequence $\{f_k\}$ that converges uniformly on each K_j and hence on each compact subset of $\mathbb{H}$.

Let $f(z) = \lim_{k\to\infty} f_k(z)$ then by Weierstrass theorem f is analytic on $\mathbb{H}$. Our aim is to show that indeed for $g(z) = f(z + b(z)) - f(z + b_0)$ we have $g \in H^{\infty}(\mathbb{H})$ or in other words for $g_k(z) = f_k(z + b(z)) - f_k(z + b_0)$, g_k converges uniformly on $\mathbb{H}$.

Let $\varepsilon > 0$ be given then since $\lim_{z\to\infty} b(z) = b_0$ we have $j_0 \in \mathbb{N}$ such that

$$| b(z) - b_0 | < \varepsilon \qquad \forall z \in \mathbb{H} \backslash M_{j_0}$$

$$M_{j_0} = \{x + iy \in \mathbb{H} : | x | \leq j_0, 0 < y \leq j_0\}$$

Now let $\alpha = \inf_{z\in\mathbb{H}} \Im(b(z))$. Since $b(\mathbb{H})$ is compact in $\mathbb{H}$ we have $\alpha > 0$. And let $S_\alpha = \{x + iy \in \mathbb{H} : y > \alpha\}$. Take $z \in S_\alpha$ and let Γ be the circle of radius α and center z. Then by Cauchy Integral Formula and Cauchy estimates on derivatives we have

$$f_k'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f_k(\zeta) d\zeta}{(\zeta - z)^2} \Longrightarrow | f_k'(z) | \leq \frac{1}{\alpha} \| f_k \|_\infty$$

hence

$$\sup_{z\in S_\alpha} | f_k'(z) | \leq \frac{1}{\alpha} \| f_k \|_\infty \leq \frac{1}{\alpha}.$$

Combining this with Mean Value Theorem we have

$$| f_k(z + b(z)) - f_k(z + b_0) | \leq \frac{1}{\alpha} | b(z) - b_0 | \qquad \forall z \in \mathbb{H}.$$

Since $b(\mathbb{H})$ is compact in $\mathbb{H}$ we have $\overline{\varphi(M_{j_0})}$ is compact in $\mathbb{H}$. So we have

$$\sup_{z\in M_{j_0}} | f_k(z + b(z)) - f_k(z + b_0) | \leq 2 \| f_k \|_\infty \leq 2 \qquad \forall k.$$

Hence for $g_k(z) = f_k(z + b(z)) - f_k(z + b_0)$ we have

$$\| g_k \| \leq \max\{2, \quad \frac{\varepsilon}{\alpha}\} \qquad \forall k.$$

So for $g(z) = \lim_{k\to\infty} g_k(z)$ we have $g \in H^\infty(\mathbb{H})$. The sequence $\{g_k\}$ converges uniformly on $\mathbb{H}$. Therefore for any sequence $\{f_n\}$ such that $\| f_n \|_\infty \leq 1$, $\{(C_\varphi - T_{b_0})f_n\}$ has a convergent subsequence in $H^\infty(\mathbb{H})$.

As a result $C_\varphi - T_{b_0}$ is compact on $H^\infty(\mathbb{H})$. $\square$

We have the following two main results, one for composition operators on $H^\infty(\mathbb{H})$ and the other for composition operators on $H^\infty(\mathbb{D})$:

Theorem C. Let $\varphi : \mathbb{H} \to \mathbb{H}$ be an analytic self-map of the upper half plane satisfying

(a) $\varphi(z) = z + b(z)$ where $b : \mathbb{H} \to \mathbb{H}$ is a bounded analytic function satisfying $\Im(b(z)) \geq M > 0$ for all $z \in \mathbb{H}$ and for some M positive,

(b) The limit $\lim_{z\to\infty} b(z) = b_0$ exists and $b_0 \in \mathbb{H}$.

Let $T_{b_0} : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$ be the translation operator $T_{b_0} f(z) = f(z + b_0)$. Then we have

$$\sigma_e(C_\varphi) = \sigma_e(T_{b_0}) = \{e^{itb_0} : t \in [0, \infty)\} \cup \{0\}$$

Theorem D. If $\varphi : \mathbb{D} \to \mathbb{D}$ is of the following form

$$\varphi(w) = \frac{2iw + b(\frac{i(1-w)}{1+w})(1-w)}{2i + b(\frac{i(1-w)}{1+w})(1-w)}$$

with $b : \mathbb{H} \to \mathbb{H}$ bounded analytic with $b(\mathbb{H}) \subset\subset \mathbb{H}$ and $\lim_{z\to\infty} b(z) = b_0$ then for $C_\varphi : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ we have

$$\sigma_e(C_\varphi) = \{e^{itb_0} : t \in [0, \infty)\} \cup \{0\}.$$

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