

Iteration of analytic self-maps of the disk: an overview ^{1 2}

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ABSTRACT

In this expository article we discuss the theory of iteration for functions analytic in the unit disk, and bounded by one in modulus. This is a classical subject that started in the eighteen-hundreds and continues today. We will review the historical background, and then conclude with more recent work.

1 Introduction

The main object of study for us is an analytic function ϕ in the unit disk $\mathbb{D} := \{|z| < 1\}$ of the complex plane, with the property that $|\phi(z)| < 1$ for all $z \in \mathbb{D}$, so that iteration becomes possible. We remind the reader that *analytic* means that near every point z_0 the function ϕ is equal to the sum of a convergent series in powers of $(z - z_0)$. In particular, ϕ can be expanded at the origin and it then takes the form

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n. \quad (1.1)$$

It is well known that the series above converge uniformly and absolutely on compact subsets of \mathbb{D} . An example, which arises in practice, of why one might be interested in iterating such functions, is the following: suppose that $0 \leq a_{n,1} < 1$ is a sequence

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of real numbers with the property that $\sum_{n=0}^{\infty} a_{n,1} = 1$, this can be interpreted as a probability mass distribution on the integers, and then $\phi(z)$ as in (1.1) is interpreted as a generating function of the distribution. Composing ϕ with itself and expanding the composite into a new power series gives rise to a new sequence $a_{n,2}$, i.e., a new probability distribution on the integers, and so on. Let now X_1 be a random variable representing the number of children that a certain self-dividing organism might provide on a given year, and suppose that the probability $P(X_1 = n)$ is equal to $a_{n,1}$. Let two years go by so that each of the children of the initial organism has also divided into more “grandchildren”, and let X_2 be the total number of grandchildren. Then $P(X_2 = n)$ is equal to $a_{n,2}$. In general, if $a_{n,N}$ are the coefficients of $\phi_N = \phi \circ \dots \circ \phi$, N iterations, and X_N is the total number of individuals at the N^{th} generation, then $P(X_N = n) = a_{n,N}$, see [GW86] Chapter 9.

Another motivation that is often given when iterating analytic functions, is its relation to Newton’s method for finding the zeros of a real polynomial. This is how the subject supposedly developed, from the real line to the complex plane, and also to several complex variables. In Newton’s method the function that is actually iterated is an analytic self-map of the Riemann sphere, i.e., a rational map. However, after dividing the plane into the *Fatou set* (good dynamics) and the *Julia set* (chaotic dynamics), it can happen that the rational map preserves a (simply-connected) component of the Fatou set and hence that it can be conjugated to a self-map of the unit disk.

Aside for the historical reasons for studying the dynamics of a self-map, there is also an inherent gain of knowledge about a given function if one knows what happens after several iterations, that is, in the “long run”. The given function is thus embedded into a “flow”, however not a continuous flow, but a discrete one where the frames of the movie are the iterates of the self-map. This usually reveals or is tied in with some hidden geometry.

As said above, in this note we are concerned with the iteration of an analytic self-map of the unit disk. More precisely, we are interested in the following problem: Suppose one picks a point $z_0 \in \mathbb{D}$ and lets $z_n = \phi_n(z_0)$, $n = 1, 2, 3, \dots$, be the associated (forward) orbit. What can be said about the convergence of such an orbit as n tends to $+\infty$ and how does it depend on z_0 ? Also, assuming that one can find preimages at each step and build the backward orbit $w_m = z_{-m}$, $m = 1, 2, 3, \dots$, so that $\phi(w_{m+1}) = w_m$ and so that $\phi(w_1) = z_0$, what can be said of the convergence of $\{w_m\}$ as m tends to $+\infty$?

2 The Denjoy-Wolff Theorem

The Denjoy-Wolff Theorem [De26] and [Wo26] provides an answer to the question of where the forward orbits go. Namely, except for the very special case of a rotation centered at the origin, (or when ϕ is an *elliptic automorphism*, i.e., it can be conjugated via a Möbius transformation of the disk to a rotation centered at the origin), in all other cases there is a special point ω in $\overline{\mathbb{D}}$, so that no matter how one chooses the initial point z_0 in \mathbb{D} , the forward orbit z_n will converge to ω . This point ω is known

as the *Denjoy-Wolff point* of the map ϕ . In the rest of this note we will assume that the self-map ϕ is not an elliptic automorphism.

There are several proofs for the Denjoy-Wolff Theorem, but they all revolve around the well-known fact (Schwarz's Lemma) that analytic self-maps of the disk are contractions with respect to the hyperbolic distance in the unit disk. In fact, this has allowed generalizations of the Denjoy-Wolff Theorem to various settings: Riemann surfaces, Riemannian manifolds, Hilbert and Banach spaces, metric spaces, etc. . . see [Be90] and [KKR99] and references therein.

Let us recall that the hyperbolic distance between two points z, w in \mathbb{D} is computed as:

$$d(z, w) = \log \frac{1 + \left| \frac{z-w}{1-\bar{w}z} \right|}{1 - \left| \frac{z-w}{1-\bar{w}z} \right|}.$$

The hyperbolic distance is invariant under Möbius transformations of the disk, the disks in this metric are Euclidean disks in shape but the centers are displaced towards $\partial\mathbb{D}$, except when they are centered at the origin; in particular the topology induced by this metric is the same as the Euclidean topology of \mathbb{D} ; and finally \mathbb{D} itself is a hyperbolic disk of infinite radius.

The Denjoy-Wolff Thm. contains more pieces of information about ω , and more is now known about the way forward orbits approach ω . All this is best described using a classification of self-maps of \mathbb{D} into three classes: elliptic, hyperbolic, and parabolic.

2.1 Elliptic maps

The first class is that of *elliptic* maps. This corresponds to the case when ω is in the open disk \mathbb{D} . It follows in this case that ω is a fixed point for ϕ , i.e., $\phi(\omega) = \omega$. It is a consequence of Schwarz's Lemma that self-maps of the disk can have at most one fixed point in \mathbb{D} and if that is the case the derivative at such fixed point (sometime called the *multiplier*) must be less or equal to 1 in modulus, because if the derivative were to be equal to 1 in modulus then ϕ would be an elliptic automorphism, again by Schwarz's Lemma, and we have excluded this special case from the beginning.

Schwarz's Lemma in this case also provides a geometric explanation. Namely, the hyperbolic disks $\Delta(\omega, r)$ with $r > 0$ form an exhaustion of \mathbb{D} as r tends to $+\infty$, and the self-map ϕ sends every disk $\Delta(\omega, r)$ into itself. Actually more is true since the multiplier at ω has modulus strictly less than 1: there is $0 < a < 1$ such that

$$\phi(\Delta(\omega, r)) \subset \Delta(\omega, ar)$$

whenever r is small enough.

There is one further distinction to be made: if $\phi'(\omega) \neq 0$, then ω is said to be an *attracting* fixed point, and if instead $\phi'(\omega) = 0$, then ω is said to be a *super-attracting* fixed point. The model self-maps for the attracting fixed point case are $\phi(z) = \lambda z$, where λ is a complex number with $0 < |\lambda| < 1$, in which case given an arbitrary initial point z_0 the corresponding forward orbit is $z_n = \lambda^n z_0$. In the super-attracting case the model maps are $\phi(z) = z^N$, where N is some integer greater or equal to two.

Here the forward orbits are $z_n = (z_0)^{N^n}$. These two types of maps are not as special as one may think. In fact, suppose ω is attracting; then one can change variables analytically in a neighborhood of ω and conjugate ϕ to the map λz , for $\lambda = \phi'(\omega)$. More precisely, there is an analytic map σ which sends a small neighborhood of ω one-to-one (conformally) onto a small neighborhood of the origin so that

$$\sigma \circ \phi \circ \sigma^{-1}(z) = \phi'(\omega)z$$

for all z near 0. Moreover, the derivative of σ at ω is equal to 1, so that σ is almost the identity near ω , and the forward orbits of ϕ really start to look like the forward orbits of λz . The existence of this map σ was proved by Kœnigs in 1884 [Koe84]. Actually, the map σ extends to a semi-conjugation defined in all of \mathbb{D} , i.e., σ is analytic in \mathbb{D} , but not necessarily one-to-one in all of \mathbb{D} , and solves the functional equation

$$\sigma \circ \phi(z) = \lambda \sigma(z).$$

Likewise in the super-attracting case Böttcher in 1905 [Bo04] produced an analytic change of variables which conjugates ϕ to z^N for some $N \geq 2$ in a neighborhood of ω .

2.2 Hyperbolic case

Here the Denjoy-Wolff point belongs to the unit circle and without loss of generality we can assume that $\omega = 1$. Once again the Denjoy-Wolff point plays the role of a fixed point for ϕ . However, the situation is more delicate here because the point 1 is not in the domain of definition of ϕ . So what we mean by fixed point is that the limit of $\phi(r)$ as $r \uparrow 1$ exists and is equal to 1. More generally, ϕ has a *non-tangential limit* 1 at 1, i.e., if θ is varying between $-\pi/2 + \epsilon$ and $\pi/2 - \epsilon$, for some $\epsilon > 0$, then $\phi(1 - te^{i\theta})$ tends to 1 as $t \downarrow 0$. The map ϕ is said to be of *hyperbolic* type if the multiplier at 1 is a number $0 < c < 1$, meaning that the non-tangential limit of ϕ' at 1 exists and is equal to $c \in (0, 1)$. The geometric interpretation is obtained from a boundary version of Schwarz's Lemma known as Julia's Lemma. To better illustrate this we introduce the Poisson kernel with pole at 1:

$$P(z) = \frac{1 - |z|^2}{|1 - z|^2}$$

which is positive and harmonic on \mathbb{D} , is equal to 0 when $|z| = 1$ and $z \neq 1$, and has non-tangential limit equal to $+\infty$ at 1. The level-sets $H(t) = \{z \in \mathbb{D} : P(z) > 1/t\}$ are disks tangent to $\partial\mathbb{D}$ at 1; they usually are referred to as *horodisks*. Notice that $\bigcap_{t>0} \overline{H(t)} = \{1\}$ and $\bigcup_{t>0} H(t) = \mathbb{D}$.

The existence of the multiplier c at 1 implies, by Julia's Lemma (see [Sh93] p. 63), that the horodisk $H(t)$ is mapped into the horodisk $H(ct)$. Julia's Lemma is a consequence of Schwarz's Lemma applied to larger and larger disks so as to exhaust a horodisk. As in the elliptic case, this is a global behavior of the map ϕ ; however, a conjugation exists in this case as well, which tells more about the behavior of a

single orbit. In 1931 Valiron [Va31] showed that there exists an analytic map σ on \mathbb{D} with $\operatorname{Re} \sigma > 0$ such that $\sigma \circ \phi = (1/c)\sigma$. Moreover, σ sends 1 to ∞ and preserves angles at 1. This fact implies that starting with an arbitrary point $z_0 \in \mathbb{D}$ the forward orbit $z_n = \phi_n(z_0)$ tends to 1 while becoming asymptotic to a ray $[1, 1 - e^{i\theta}]$ for some $\theta \in (-\pi/2, \pi/2)$.

2.3 Parabolic case

Finally, the remaining case is that the Denjoy-Wolff point is on $\partial\mathbb{D}$ as in the hyperbolic case, say at 1, but that the multiplier at 1 is now equal to 1. This is the most delicate and richest case of the three. Since 1 is the point of interest, it is best to change variables from \mathbb{D} to the upper half-plane \mathbb{H} , with the map $i(1+z)/(1-z)$ and its inverse, so that 1 goes to ∞ . In this set-up, ϕ is an analytic map on the upper half-plane \mathbb{H} , with $\operatorname{Im} \phi(z) \geq 0$, such that

$$\phi(z) \longrightarrow \infty \quad \text{and} \quad \frac{\phi(z)}{z} \longrightarrow 1. \quad (2.1)$$

as $z \rightarrow \infty$ non-tangentially. The non-tangential approach regions for ∞ in \mathbb{H} are the sectors $\{\theta_0 < \operatorname{Arg} z < \pi - \theta_0\}$ for some $\theta_0 \in (0, \pi/2)$. Since the horodisks at infinity are the half-planes $\{\operatorname{Im} z \geq t > 0\}$, Julia's Lemma in this situation implies that $\operatorname{Im} \phi(z) \geq \operatorname{Im} z$. In \mathbb{H} the hyperbolic distance between two points z, w is:

$$d = d(z, w) = \log \frac{1 + \left| \frac{w-z}{w-\bar{z}} \right|}{1 - \left| \frac{w-z}{w-\bar{z}} \right|}$$

Let $z_n = \phi_n(i)$. We know that z_n tends to infinity by the Denjoy-Wolff Theorem. Also the step-lengths $s_n = d(z_n, z_{n+1})$ decrease to some limit s_∞ , and ϕ is said to be of type I (or non-zero-step) if $s_\infty > 0$; ϕ is of type II (or zero-step) if $s_\infty = 0$. By Julia's Lemma, $\operatorname{Im} z_n \uparrow L_\infty$ for some limit $0 < L_\infty \leq \infty$. A map ϕ of type I is said to be of **type Ia** (or non-zero-step/finite-height) if $L_\infty < \infty$ and of **type Ib** (or non-zero-step/infinite-height) if $L_\infty = \infty$. Likewise for type II.

Example 2.1 *The map $\phi(z) = z + 1$ is of type Ia, while $\phi(z) = z + i$ is of type Ib.*

It is not clear *a priori* that this classification does not depend on the choice of i as starting point, however, that is indeed the case, as a consequence of the next theorem.

Theorem 2.2 (Pommerenke, [Pom79] (3.17)) *Let ϕ be an analytic self-map of \mathbb{H} of parabolic type as in (2.1), and let $\{z_n = \phi_n(i)\}_{n=0}^\infty$ be a forward-iteration sequence. Then*

$$\frac{\operatorname{Im} z_{n+1}}{\operatorname{Im} z_n} \longrightarrow 1$$

as n tends to infinity.

Moreover, letting $z_n = u_n + iv_n$ and considering the automorphisms of \mathbb{H} given by $M_n(z) = (z - u_n)/v_n$, the normalized iterates $M_n \circ \phi_n$ converge uniformly on compact subsets of \mathbb{H} to a function σ with $\text{Im } \sigma > 0$, which satisfies the functional equation

$$\sigma \circ \phi = \sigma + b \tag{2.2}$$

where

$$b = \lim_{n \rightarrow \infty} \frac{u_{n+1} - u_n}{v_n} \tag{2.3}$$

and $b \neq 0$ in the non-zero-step case, while $b = 0$ in the zero-step case.

Pommerenke's conjugation is not useful in the zero-step case, so in [BP79], a different conjugation is obtained in this case, as in (2.2), but without the restriction $\text{Im } \sigma > 0$.

A different approach to these conjugations results is contained in [Co81], see also [CM95].

3 Backward-iteration

The recent advances in the iteration of self-maps of the disk have been spurred by the theory of composition operators, see [Sh93] and [CM95]. Every analytic self-map of the disk naturally induces a linear operator $C_\phi(f) = f \circ \phi$ on the space of all analytic functions f defined on \mathbb{D} . The operator C_ϕ happens to preserve many of the classical Hilbert and Banach spaces of analytic functions on \mathbb{D} , and the spectral theory of C_ϕ is intimately tied in with the behavior of the iterates ϕ_n near $\partial\mathbb{D}$. See for instance Theorem 3.1 of [PC98] and its corollaries; see also Theorem 7.23 of [CM95].

We finish this survey by quickly reviewing some results in [PC00] and [PC]. Recall that a backward orbit $w_m = z_{-m}$, $m = 1, 2, 3, \dots$, is such that $\phi(w_{m+1}) = w_m$ and $\phi(w_1) = z_0$. There are simple maps, e.g., $z/2$, with no backward orbits whatsoever, and there are also maps with many backward orbits with very complicated behaviors, e.g., inner functions.

We need a “normalizing” concept: the essential tool used in the forward case was the fact that self-maps of the disk are contractions in the hyperbolic metric, so in particular the hyperbolic steps between two consecutive elements of a forward orbit are always decreasing. On the other hand, the steps between two consecutive elements of a backward orbit are always increasing. The following observation turns out to be very helpful: Let us consider backward orbits $\{w_m\}$ with bounded hyperbolic steps, i.e., so that the distance between two consecutive terms of the sequence in the hyperbolic metric is bounded above by some constant $C < \infty$. We call such sequences BISBS as in Backward Iteration Sequences of Bounded Step. These sequences enjoy several nice properties that are reminiscent of the forward case. First of all, a BISBS always converges to a boundary point $\zeta \in \partial\mathbb{D}$. Here two cases arise: either $\zeta = \omega$ is the Denjoy-Wolff point, in which case, the map ϕ is necessarily of parabolic type, see section 2.3; or $\zeta \neq \omega$, in which case ζ is a so-called Boundary Repelling Fixed Point (BRFP), i.e., ϕ has non-tangential limit ζ at ζ , and the multiplier A of ϕ at ζ exists in the sense of non-tangential limits and satisfies $1 < A < \infty$. Moreover, when ζ is a

BRFP, the BISBS w_m tends to ζ while becoming asymptotic to a ray $[\zeta, \zeta(1 - e^{1\theta})]$ for some $\theta \in (-\pi/2, \pi/2)$, in total similarity with the forward hyperbolic case of Section 2.2. Conversely, if ζ is a BRFP, then there are many BISBS converging to ζ in all possible directions θ and there is also a conjugation available; for more details we refer to Theorem 1.2 and Corollary 1.5 of [PC00].

Similarly to the forward iterations the richest behavior is when the BISBS tends to ω in the parabolic case. There are several questions still open in this situation; however something can be said. Let's assume again that ϕ is an analytic map on the upper half-plane \mathbb{H} , with $\text{Im} \phi(z) \geq 0$, as in (2.1). By Julia's Lemma, $y_n := \text{Im}(w_n) \downarrow \ell_\infty$ for some $0 \leq \ell_\infty < \infty$. We say that a BISBS is of **type 1** (or non-zero-height) if $\ell_\infty > 0$, and of **type 2** (or zero-height) if $\ell_\infty = 0$. So, for instance, $\phi(z) = z + 1$ has a BISBS of type 1. At first sight, one might think that type 2 never occurs. However, a simple example is given by the following map $\phi(z) = \sqrt{z^2 - 1}$. Thinking of ϕ as the composition of three simple operations, one checks that ϕ maps \mathbb{H} into itself and that it is of parabolic type IIb with Denjoy-Wolff point at infinity. The sequence $w_n = \sqrt{n + i}$ is a backward-iteration sequence for ϕ . A calculation shows that the hyperbolic steps stay bounded away from infinity. So w_n is a BISBS and $y_n = \text{Im} w_n$ is asymptotic to $1/\sqrt{n}$, which tends to 0. In this example, although y_n tends to 0, it does not do so very fast, i.e., $\sum y_n = \infty$. This turns out to be always the case.

Theorem 3.1 ([PC]) *Let ϕ be an analytic self-map of \mathbb{H} of parabolic type as in (2.1), and let $\{w_n\}_{n=0}^\infty$ be a backward-iteration sequence with bounded hyperbolic steps $d_n = d(w_n, w_{n+1}) \uparrow C < \infty$, such that $w_n \rightarrow \infty$. Then*

$$\frac{\text{Im} w_{n+1}}{\text{Im} w_n} \longrightarrow 1$$

as n tends to infinity.

This is the key to proving the following conjugation result.

Theorem 3.2 ([PC]) *Let ϕ be an analytic self-map of \mathbb{H} of parabolic type as in (2.1), and let $\{w_n = x_n + iy_n\}_{n=0}^\infty$ be a backward-iteration sequence with bounded hyperbolic steps $d_n = d(w_n, w_{n+1}) \uparrow C < \infty$, such that $w_n \rightarrow \infty$. (Assume also WLOG that $\text{Arg} w_n$ tends to 0). Consider the automorphisms of \mathbb{H} given by $\tau_n(z) = x_n + y_n z$. Then the normalized iterates $\phi_n \circ \tau_n$ converge uniformly on compact subsets of \mathbb{H} to an analytic self-map ψ of \mathbb{H} such that*

$$\psi(z - b_0) = \phi \circ \psi(z)$$

where

$$b_0 = \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{y_n} > 0$$

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