conven...,

$$\langle \cdots, \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \xi_2, \cdots \rangle;$$

the term in parentheses indicates the one corresponding to the index 0. The bilateral shift is the operator W on $\mathbf{H}$ defined by

$$W\langle \cdots, \xi_{-2}, \xi_{-1}, (\xi_0), \xi_1, \xi_2, \cdots \rangle = \langle \cdots, \xi_{-3}, \xi_{-2}, (\xi_{-1}), \xi_0, \xi_1, \cdots \rangle.$$

Linearity is obvious, and boundedness is true with room to spare; the bilateral shift, like the unilateral one, is an isometry. Since the range of the bilateral shift is the entire space $\mathbf{H}$, it is even unitary.

If e_n is the vector $\langle \cdots, \xi_{-1}, (\xi_0), \xi_1, \cdots \rangle$ for which $\xi_n = 1$ and $\xi_i = 0$ whenever $i \neq n$ ($n = 0, \pm 1, \pm 2, \cdots$), then the e_n's form an orthonormal basis for $\mathbf{H}$. The effect of W on this basis is described by

$$We_n = e_{n+1} \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Problem 68. *What is the spectrum of the bilateral shift, and what are its parts (point spectrum, compression spectrum, and approximate point spectrum)? What are the answers to the same questions for the adjoint of the bilateral shift?*

69. Spectrum of a functional multiplication. Every operator studied so far has been a multiplication, either in the legitimate sense (on an $\mathbf{L}^2$) or in the extended sense (on a functional Hilbert space). The latter kind is usually harder to study; it does, however, have the advantage of having a satisfactory characterization in terms of its spectrum.

Problem 69. *A necessary and sufficient condition that an operator A on a Hilbert space* $\mathbf{H}$ *be representable as a multiplication on a functional Hilbert space is that the eigenvectors of A^* span* $\mathbf{H}$.

Caution: as the facts for multiplications on $\mathbf{L}^2$ spaces show (cf. Solution 66) this characterization is applicable to functional Hilbert spaces only. The result seems to be due to P. R. Halmos and A. L. Shields.

70. Relative spectrum of shift. An operator A is *relatively inver*[...] if there exists an operator B such that $ABA = A$. This is a ra[...] special concept, not particularly useful, but with some curious proper[...] Clearly every invertible operator is relatively invertible; in fact ev[...] operator that is either left invertible or right invertible is also relati[...] invertible. These remarks are obvious; it is much less obvious (but tr[...] that every operator on a finite-dimensional space is relatively inverti[...] (Hint: write the operator as a direct sum of an invertible operator [...] a nilpotent one.) The concept belongs to general ring theory; the [...]sertion about finite-dimensional spaces can be expressed by saying t[...] a finite-dimensional full matrix algebra over the complex numbers [...] *regular ring* (see von Neumann [1936]). The *relative spectrum* of [...] operator A (on a Hilbert space of any dimension) is the set of all th[...] complex numbers λ for which $A - \lambda$ is not relatively invertible.

Problem 70. *What is the relative spectrum of the unilateral shift?*

The concept of relative spectrum was introduced and studied [...] Asplund [1958].

71. Closure of relative spectrum.

Problem 71. *Is the relative spectrum always closed?*

HINTS

Problem 69. Use a spanning set of eigenvectors of A^* for the domain; for each f in that domain, define the multiplier as the conjugate of the corresponding eigenvalue.

for all f, so that $U - \lambda$ is bounded from below; this proves that $\Pi(U)$ is exactly the unit circle. For U^* the situation is different: since Π_0 is always included in Π, and since $\Pi_0(U^*)$ is the open unit disc, it follows that $\Pi(U^*)$ is the closed unit disc.

Solution 68. *If W is the bilateral shift, then $\Lambda(W) = C$ (= the unit circle), $\Pi_0(W) = \varnothing$, $\Pi(W) = C$, and $\Gamma(W) = \varnothing$. The same equations are true for the adjoint W^*.*

Proof. The determination of the spectrum of W, and of the fine structure of that spectrum, can follow the pattern indicated in the study of the unilateral shift U (Solution 67), but there is also another way to do it, a better way. Corresponding to the functional representation of U on $\mathbf{H}^2$, the bilateral shift W has a natural functional representation on $\mathbf{L}^2(\mu)$ (where μ is normalized Lebesgue measure on the unit circle; see Problem 26). Since the functions e_n defined by $e_n(z) = z^n$ ($n = 0, \pm 1, \pm 2, \cdots$) form an orthonormal basis for $\mathbf{L}^2$, and since the effect on them of shifting forward by one index is the same as the effect of multiplying by e_1, it follows that the bilateral shift is the same as the multiplication operator on $\mathbf{L}^2$ defined by

$$(Wf)(z) = zf(z).$$

This settles everything for W; everything follows from Solution 66.

As for W^*, its study can be reduced to that of W. Indeed, since W is unitary, its adjoint is the same as its inverse. The calculation of W^{-1} takes no effort at all; clearly W^{-1} shifts backward the same way as W shifts forward. There is a thoroughgoing symmetry between W and W^*; to obtain one from the other, just replace n by $-n$. In more pedantic language: W and W^* are unitarily equivalent, and, in particular, the unitary operator R determined by the conditions $Re_n = e_{-n}$ ($n = 0, \pm 1, \pm 2, \cdots$) transforms W onto W^* (i.e., $R^{-1}WR = W^*$). Conclusion: the spectrum of W^* is equal to the spectrum of W, and the same is true, part for part, for each of the usual parts of the spectrum.

Solution 69. Suppose first that the eigenvectors of A^* span $\mathbf{H}$. Let X be an index set such that corresponding to each x in X there is an

eigenvector K_x of A^*, and such that the K_x's span $\mathbf{H}$; denote the eigenvalue corresponding to K_x by $\varphi(x)^*$. (The conjugation has no profound significance here; it is just a notational convenience.) It follows that $A^*K_x = \varphi(x)^*K_x$. For each f in $\mathbf{H}$, let $\tilde{f}$ be the function on X defined by $\tilde{f}(x) = (f,K_x)$. The correspondence $f \to \tilde{f}$ is linear. If $\tilde{f} = 0$, i.e., if $(f,K_x) = 0$ for all x, then $f = 0$ (since the K_x's span $\mathbf{H}$). This justifies the definition $(\tilde{f},\tilde{g}) = (f,g)$. With this definition of inner product, the set $\tilde{\mathbf{H}}$ of all functions of the form $\tilde{f}$ (with f in $\mathbf{H}$) becomes a functional Hilbert space. [Note: $|\tilde{f}(x)| = |(f,K_x)| \leqq \|f\|\cdot\|K_x\| = \|\tilde{f}\|\cdot\|K_x\|$.] Let $\tilde{A}$ be the image of A under the isomorphism $f \to \tilde{f}$ (i.e., $\tilde{A}\tilde{f} = (Af)^{\sim}$); then

$$(\tilde{A}\tilde{f})(x) = (Af)^{\sim}(x) = (Af,K_x) = (f,A^*K_x)$$

$$= (f,\varphi(x)^*K_x) = \varphi(x)(f,K_x)$$

$$= \varphi(x)\tilde{f}(x),$$

so that $\tilde{A}$ is a multiplication.

The converse is proved by retracing the steps of the last computation. In detail, if A is a multiplication (with multiplier φ, say) on a functional Hilbert space $\mathbf{H}$ with domain X and kernel function K, so that $(Af)(x) = \varphi(x)f(x)$, then $(Af,K_x) = \varphi(x)(f,K_x)$ (where $K_x(y) = K(y,x)$), and therefore $(f,A^*K_x - \varphi(x)^*K_x) = 0$ for all f. It follows that $A^*K_x = \varphi(x)^*K_x$; since in a functional Hilbert space the set of K_x's always spans the space, the proof is complete.

Compare the construction with what is known about the unilateral shift (Solution 67).

Solution 70. *The relative spectrum of the unilateral shift is the unit circle.*

Proof. The proof can be made to depend on two simple lemmas. (1) For an operator with a trivial kernel, relative invertibility is the same as left invertibility. (2) For all operators, left invertibility is the same as boundedness from below.

The proof of (1) in one direction is trivial; left invertibility always implies relative invertibility. To prove the converse, suppose that