10. (page 86 of Bartle and Sherbert)

Let $x_1 < x_2$ be real numbers and define the sequence (x_n) recursively by

$$x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$$

for $n > 2$. Show that the sequence (x_n) converges and find its limit.

Solution:

We will solve this problem by a series of lemmas, some of which are more general than the problem itself. The solution given does not directly use the Contraction Mapping Theorem (Theorem 3.5.8) but is clearly closely related to it.

The first lemma shows that the between-ness that follows from the averaging process gives an order relation among the terms of the sequence (x_n) that is helpful in the analysis.

Lemma 1. *Suppose (x_n) is a sequence with x_{k+1} (strictly) between x_k and x_{k-1} for each integer k with $k \geq 2$ and $x_1 < x_2$. Then*

$$x_1 < x_3 < x_5 < x_7 < \cdots < x_8 < x_6 < x_4 < x_2 \tag{1}$$

that is, the sequence satisfies $x_{2k-1} < x_{2k+1} < x_{2j+2} < x_{2j}$ for each j and k in $\mathbb{N}$.

Proof. We are given that $x_1 < x_2$. By hypothesis, with $k = 2$, we know x_3 is between x_2 and x_1, so $x_1 < x_3 < x_2$. Now, for $k = 3$, we have x_4 is between x_3 and x_2, so $x_1 < x_3 < x_4 < x_2$.

Thus, for $n = 1$, we have $x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n}$. We establish this inequality for all n by induction. Suppose this inequality is true for $n = \ell \geq 1$, that is, $x_{2\ell-1} < x_{2\ell+1} < x_{2\ell+2} < x_{2\ell}$, then we want to show it is true for $n = \ell + 1$. For $k = 2\ell + 2$, we have $x_{2\ell+1} = x_{k-1} < x_k = x_{2\ell+2}$ and by hypothesis, x_{k+1} is between x_k and x_{k-1}, that is, $x_{k-1} < x_{k+1} < x_k$, and again, x_{k+2} is between x_{k+1} and x_k, so we have $x_{k-1} < x_{k+1} < x_{k+2} < x_k$. Recalling that $k = 2\ell + 2$, this means $x_{2\ell+1} < x_{2\ell+3} < x_{2\ell+4} < x_{2\ell+2}$ which is the desired inequality $x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n}$ for $n = \ell + 1$.

By induction, this means that $x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n}$ is true for all positive integers n. Now, suppose k and j are integers in $\mathbb{N}$. If $k = j$, then this is just one instance of the inequality we have proved. If $k < j$, then, for $n = j$, we have $x_{2j-1} < x_{2j+1} < x_{2j+2} < x_{2j}$. But since the inequalities on the left hold for all n, we have

$$x_{2k-1} < x_{2k+1} < \cdots < x_{2j+1} < x_{2j+2} < x_{2j}$$

and the inequality in the conclusion holds. Similarly, if $k > j$, then for $n = k$, we have $x_{2k-1} < x_{2k+1} < x_{2k+2} < x_{2k}$ and since the inequalities on the right hold for all n, we have

$$x_{2k-1} < x_{2k+1} < x_{2k+2} < \cdots < x_{2j+2} < x_{2j}$$

and the Lemma is proved for all k and j in $\mathbb{N}$. □

The second lemma shows that the order relations from Lemma 1, together with a hypothesis on successive terms being close, imply convergence of the sequence.

Lemma 2. *Suppose the sequence (y_n) satisfies Inequality (1), that is, it satisfies $y_{2k-1} \leq y_{2k+1} \leq y_{2j+2} \leq y_{2j}$ for each j and k in $\mathbb{N}$. Then the sequence (y_n) converges if and only if $\lim_{n\to\infty} |y_n - y_{n+1}| = 0$.*

Proof. If the sequence (y_n) converges, then the sequence (y_n) is Cauchy, and for any $\epsilon > 0$, there is N_ϵ so that when n and m satisfy $n, m > N_\epsilon$, then $|y_n - y_m| < \epsilon$. Choosing, $m = n + 1$, this means when $n > N_\epsilon$, then $|y_n - y_{n+1}| < \epsilon$. This means that $\lim_{n\to\infty} |y_n - y_{n+1}| = 0$.

Conversely, if (y_n) satisfies Inequality (1), then the sequence (y_{2k-1}) is an increasing sequence and the sequence (y_{2j}) is a decreasing sequence. We will see that this means that (y_n) is a Cauchy sequence. If $\epsilon > 0$ is given, then since $\lim_{n\to\infty} |y_n - y_{n+1}| = 0$, there is M_ϵ so that $n > M_\epsilon$ implies $|y_n - y_{n+1}| < \epsilon$. Choose k so that $2k - 1 > M_\epsilon$ and let $j = k$. Then, we have $(2k-1)+1 = 2k = 2j$ and we see $|y_{2k-1} - y_{2j}| < \epsilon$. On the other hand, if n and m satisfy $n, m > 2k = 2j$, then y_n satisfies $y_{2k-1} < y_n < y_{2j}$ and y_m also satisfies $y_{2k-1} < y_m < y_{2j}$ so we have $|y_n - y_m| < |y_{2k-1} - y_{2j}| < \epsilon$. Since this is true for all $\epsilon > 0$, this means that (y_n) is a Cauchy sequence, and therefore (y_n) converges. □

The following lemma shows that we can normalize the sequence and thereby simplify our calculations.

Lemma 3. *Let the sequence (a_n) be defined recursively by $a_1 = 0$ and $a_1 = 1$ and*

$$a_n = \frac{1}{2}(a_{n-2} + a_{n-1})$$

for $n > 2$. Then the sequence (x_n) defined above satisfies $x_n = x_1 + a_n(x_2 - x_1)$.

Proof. Notice that the hypotheses on the sequence (a_n) give

$$x_1 + a_1(x_2 - x_1) = x_1 + 0(x_2 - x_1) = x_1$$

and

$$x_1 + a_2(x_2 - x_1) = x_1 + 1(x_2 - x_1) = x_2$$

so the formula works for $n = 1$ and $n = 2$. We continue by using (strong) induction. Suppose, $x_n = x_1 + a_n(x_2 - x_1)$ for all $n \le k$. Then

$$\begin{aligned} x_{k+1} &= \frac{1}{2}(x_k + x_{k-1}) = \frac{1}{2}(x_1 + a_k(x_2 - x_1) + x_1 + a_{k-1}(x_2 - x_1)) \\ &= x_1 + \frac{1}{2}(a_k + a_{k-1})(x_2 - x_1) = x_1 + a_{k+1}(x_2 - x_1) \end{aligned}$$

and the formula works for $n = k + 1$. Thus, the formula holds for all n in $\mathbb{N}$. □

Lemma 2 says that to establish convergence, we need to look at the differences of successive terms.

Lemma 4. *Suppose (a_n) is the sequence of Lemma 3. Then for $n \ge 2$, we have $|a_{n+1} - a_n| = |a_n - a_{n-1}|/2$ and the sequence (a_n) is convergent.*

Proof. For any $n \ge 2$, we have

$$|a_{n+1} - a_n| = |\frac{1}{2}(a_{n-1} + a_n) - a_n| = |\frac{1}{2}a_{n-1} + \frac{1}{2}a_n - a_n| = |\frac{1}{2}a_{n-1} - \frac{1}{2}a_n| = \frac{1}{2}|a_{n-1} - a_n|$$

so the desired relationship holds.

We will use induction to show that for $n \ge 2$, we have $|a_n - a_{n-1}| = 1/2^{n-2}$. For $n = 2$, we see that $|a_2 - a_1| = |1 - 0| = 1 = 1/2^0$, so the formula is correct for $n = 2$. Now suppose, we have the formula is correct for $n = k \ge 2$, that is, we have $|a_k - a_{k-1}| = 1/2^{k-2}$. Then, using the relationship just proved, we have

$$|a_{k+1} - a_k| = \frac{1}{2}|a_k - a_{k-1}| = \frac{1}{2}\frac{1}{2^{k-2}} = \frac{1}{2^{k-1}}$$

and the formula is correct for $n = k + 1$. Thus, the formula is correct for all $n \ge 2$.

Now, $\lim_{n\to\infty} 1/2^{n-1} = 0$, so this says that $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$. Now the sequence (a_n) satisfies the hypotheses of both Lemma 1 and and Lemma 2, so this means that the sequence (a_n) converges. □

The following establishes a relationship that is useful in finding the limit of the sequence (a_n).

Lemma 5. *Suppose (a_n) is the sequence of Lemma 3. Then for $k = 3, 4, 5, \cdots$, we have $a_{k+2} - a_k = (a_k - a_{k-2})/4$.*

Proof. Suppose (a_n) is the sequence above. Then for $k = 3, 4, 5, \cdots$, we have $|a_k - a_{k-1}| = |a_{k-1} - a_{k-2}|/2$. We know from Lemma 1 that the even numbered terms of the sequence are larger than the odd numbered terms, so being aware of the parity of k, we have $a_k - a_{k-1} = (a_{k-2} - a_{k-1})/2$ where both sides are positive if k is even and both sides negative if k is odd. We also have $a_{k+1} - a_k = (a_{k-1} - a_k)/2$. Adding these two equations, we get $a_{k+1} - a_{k-1} = (a_{k-2} - a_k)/2$ for every $k \geq 3$.

Now replacing k by $k+1$, we get $a_{k+2} - a_k = (a_{k-1} - a_{k+1})/2$. Combining this with the equality in the preceding paragraph, we get

$$a_{k+2} - a_k = \frac{1}{2}(a_{k-1} - a_{k+1}) = \frac{1}{2}\left(\frac{1}{2}\right)(a_k - a_{k-2}) = \frac{1}{4}(a_k - a_{k-2})$$

as we wished to prove. □

Finally, we can put these results together to get the answer to question 10, page 86.

Theorem 6. *If $x_1 < x_2$ are real numbers and the sequence (x_n) is defined recursively by*

$$x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$$

for $n > 2$, then (x_n) converges and

$$\lim_{n \to \infty} x_n = \frac{1}{3}x_1 + \frac{2}{3}x_2$$

Proof. Lemma 3 allows us to consider the normalized sequence (a_n) rather than the given sequence (x_n). This should make the calculations more transparent.

The formula in Lemma 5, for $k \geq 3$, gives

$$a_{k+2} - a_k = \frac{1}{4}(a_k - a_{k-2})$$

Writing $k = 2j + 1$, we have, for $j \geq 1$

$$a_{2j+3} - a_{2j+1} = \frac{1}{4}(a_{2j+1} - a_{2j-1})$$

Now for $j = 1$, this says that $a_5 - a_3 = (a_3 - a_1)/4$. Since $a_1 = 0$ and $a_2 = 1$, we have $a_3 = (1 + 0)/2 = 1/2$ and, therefore, $a_5 - a_3 = 1/8$.

Now, an easy induction argument gives $a_{2j+3} - a_{2j+1} = 2^{-(2j+1)}$: It is true for $j = 1$. If it is true for $j = k$, then

$$a_{2(k+1)+3} - a_{2(k+1)+1} = \frac{1}{4}(a_{2k+3} - a_{2k+1}) = 2^{-2}2^{-(2k+1)} = 2^{-(2k+3)}$$

and it is true for $j = k + 1$, so it is true for all j in $\mathbb{N}$.

Now for k a positive integer, we have

$$a_{2k+3} = (a_{2k+3} - a_{2k+1}) + (a_{2k+1} - a_{2k-1}) + \cdots + (a_5 - a_3) + a_3 = \frac{1}{2^{2k+1}} + \frac{1}{2^{2k-1}} + \cdots + \frac{1}{2^3} + \frac{1}{2}$$

Thus,

$$\lim_{k \to \infty} a_{2k+3} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots = \frac{1}{2}(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots) = \frac{1}{2}\left(\frac{1}{1 - (1/4)}\right) = \frac{2}{3}$$

Since the sequence (a_n) converges and this subsequence converges to $2/3$, we must have $\lim_{n\to\infty} a_n = 2/3$.

Now, Lemma 3 says that $x_n = x_1 + a_n(x_2 - x_1)$, so

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty} (x_1 + a_n(x_2 - x_1)) = x_1 + \left(\lim_{n\to\infty} a_n\right)(x_2 - x_1) \\ &= x_1 + \frac{2}{3}(x_2 - x_1) = \frac{1}{3}x_1 + \frac{2}{3}x_2\end{aligned}$$

as we were to prove. □