# Questions from Old Second Tests for Math 51100

The following matrix is used in Problems 1 AND 2 AND 3.

Let $A = \begin{pmatrix} 1 & 0 & 1 & -1 & -1 \\ 2 & -1 & 1 & 0 & 0 \\ -2 & 2 & -1 & 2 & 1 \\ 3 & 2 & 3 & 1 & -1 \end{pmatrix}$

1. There are no solutions to the system

$$\begin{cases} a \phantom{+2b} + c - d - e = -3 \\ 2a - b + c = 1 \\ -2a + 2b - c + 2d + e = 2 \\ 3a + 2b + 3c + d - e = -1 \end{cases}$$

(a) Explain why, mentioning ranges of matrices, this system has no solutions.
(b) Do a calculation that justifies your assertion in (a), that is, report a calculation that your machine has done, or show a calculation that shows the assertion in (a) is correct.

(c) Find the 'least squares solution' of this system.

2. In EACH of the following EXPLAIN THE WORK YOU DID TO GET YOUR ANSWER!

(a) Is the vector $p = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}$ in $\mathcal{N}(A)$, the null space of $A$?

(b) Is the vector $q = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ in $\mathcal{R}(A)$, the range of $A$?

(c) Find an orthonormal basis for the orthogonal complement of the range of $A$, that is, find an orthonormal basis for $\mathcal{R}(A)^{\perp}$.

3. Find $w$ in $\mathcal{N}(A)$, the null space of $A$, and $u$ in $\mathcal{N}(A)^{\perp}$,

the orthogonal complement of the null space of $A$, so that $w + u = x$ where $x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

4. Vectors $u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, $u_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ are an orthogonal set of vectors in $\mathbb{R}^4$.

(a) Explain how you can easily tell that these vectors form a basis for $\mathbb{R}^4$.

(b) Write the vector $w = \begin{pmatrix} 3 \\ 2 \\ -1 \\ 4 \end{pmatrix}$ as a linear combination of $u_1$, $u_2$, $u_3$, and $u_4$.

(Explain how you found this answer.)

The following matrix is used in Problems 5 AND 6.

Let $B = \begin{pmatrix} -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 1 & 2 & -1 & 2 & -2 \\ -1 & 1 & 3 & 2 & 3 \end{pmatrix}$

5. (a) Explain how you know that the vector $v = \begin{pmatrix} -3 \\ 1 \\ 2 \\ -1 \end{pmatrix}$ is *not* in the range of $B$.

   (b) Find the distance from $v$ to the range of $B$.

6. Find $w$ in $\mathcal{N}(B)$, the null space of $B$, and $u$ in $\mathcal{N}(B)^{\perp}$,

   the orthogonal complement of the null space of $B$, so that $w + u = x$ where $x = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 3 \\ -1 \end{pmatrix}$.

The following matrix is used in Problems 7 AND 8.

Let $C = \begin{pmatrix} 1 & 2 & -1 & 2 & -2 \\ -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ -1 & 1 & 3 & 2 & 3 \end{pmatrix}$

7. (a) Explain how you know that the vector $v = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 1 \end{pmatrix}$ is *not* in the range of $C$.

   (b) Find the distance from $v$ to the range of $C$.

8. Find $w$ in $\mathcal{N}(C)$, the null space of $C$, and $u$ in $\mathcal{N}(C)^{\perp}$,

   the orthogonal complement of the null space of $C$, so that $w + u = x$ where $x = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -3 \\ 1 \end{pmatrix}$.

The following matrix is used in Problems 9 AND 10 AND 11: $D = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 2 & 2 & 2 \\ 1 & -1 & 2 & 0 \\ -1 & 2 & -1 & 1 \\ -1 & -1 & -4 & -2 \end{pmatrix}$

9. (a) What is the relationship between the range of the matrix $D$ and solutions of systems equations whose coefficient matrix is $D$.

   (b) Is $b = \begin{pmatrix} 2 \\ 5 \\ 4 \\ 1 \\ -2 \end{pmatrix}$ in $\mathcal{R}(D)$, the range of $D$?

   Do a calculation that justifies your assertion, that is, report a calculation that your machine has done, or give a calculation that shows the assertion is correct.

   (c) Find the 'least squares solution' of the system $DX = b$.

10. In EACH of the following EXPLAIN THE WORK YOU DID TO GET YOUR ANSWER!

(a) Is the vector $p = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ in $\mathcal{N}(D)$, the null space of $D$?

(b) Is the vector $q = \begin{pmatrix} -1 \\ 1 \\ -2 \\ 3 \\ -1 \end{pmatrix}$ in $\mathcal{R}(D)$, the range of $D$?

(c) Find an orthonormal basis for the range of $D$, that is, find an orthonormal basis for $\mathcal{R}(D)$.

11. Find $w$ in $\mathcal{R}(D)$, the range of $D$, and $u$ in $\mathcal{R}(D)^\perp$,

the orthogonal complement of the range of $D$, so that $w + u = x$ where $x = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

12. Let $M$ be the subspace of $\mathbf{R}^5$

spanned by $y_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$, $y_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, $y_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $y_4 = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \\ 2 \end{pmatrix}$.

Find an orthonormal basis for $M^\perp$.

13. Let $\{h_1, h_2, h_3\}$ be a set of orthonormal vectors in $\mathbb{R}^4$ and let $\mathcal{U}$ be the subspace of $\mathbb{R}^4$ that they span. Let $H$ be the matrix with columns $h_1$, $h_2$, and $h_3$, that is, in block form, $H = [h_1\ h_2\ h_3]$, and let $R = HH'$. With this construction, the range of $R$ is $\mathcal{U}$ (you may use this fact in the questions below).

(a) Show that $R = HH'$ is the orthogonal projection of $\mathbb{R}^4$ onto $\mathcal{U}$.

(b) For the vectors $\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, $\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$, and $\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$, construct the matrix $R$ as defined above.

(c) Use this matrix $R$ to find the point $u$ in $\mathcal{U}$ that is closest to the vector $z = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -2 \end{pmatrix}$.

14. Let $x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $y = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, and $z = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ be vectors in $\mathbb{R}^3$.

Find the vector $p$ in $\mathbb{R}^3$ for which the inner products of $p$ with $x$, $y$, and $z$ are $\langle x, p\rangle = 2$, $\langle y, p\rangle = -2$, and $\langle z, p\rangle = 3$.

15. Find an orthonormal basis for the subspace of $\mathbf{R}^5$ spanned by $u_1 = (1, 0, -1, 1, 1)$, $u_2 = (2, -1, -1, 0, 1)$, $u_3 = (1, 1, 0, 2, -1)$, and $u_4 = (2, -2, -2, -1, 3)$.

16. The matrix $C$ is a $7 \times 10$ matrix and the rank of $C$ is 4.

   (a) What is the dimension of $\mathcal{R}(C)$, the range of $C$?__________

   (b) What is the dimension of $\mathcal{R}(C')$, the range of $C'$?__________

   (c) What is the dimension of $\mathcal{R}(C)^{\perp}$,
   the orthogonal complement of the range of $C$?__________

   (d) What is the dimension of $\mathcal{R}(C')^{\perp}$,
   the orthogonal complement of the range of $C'$?__________

   (e) What is the dimension of $\mathcal{N}(C')^{\perp}$,
   the orthogonal complement of the null space of $C'$?__________

17. Suppose $M$ is a 5 dimensional subspace of $\mathbb{R}^8$.

   (a) What is the dimension of $M^{\perp}$, the orthogonal complement of $M$?__________

   (b) What is the dimension of $(M^{\perp})^{\perp}$,
   the orthogonal complement of the orthogonal complement of $M$?__________

   (c) What is the dimension of $M \cap M^{\perp}$,
   the intersection of $M$ and its orthogonal complement?__________

18. The vectors $u$, $v$, $w$, and $z$ are in $\mathbb{R}^4$. The following information is given about the positions of these vectors in $\mathbb{R}^4$. We have

$$\|u\| = \sqrt{3} \quad \|v\| = 2 \quad \|w\| = 3 \quad \|z\| = 2\sqrt{3}$$

$$\langle u, v\rangle = -1 \quad \langle u, w\rangle = 3 \quad \langle u, z\rangle = 5 \quad \langle v, w\rangle = -3 \quad \langle v, z\rangle = 2 \quad \langle w, z\rangle = 3$$

ONE of the vectors $w$ and $z$ IS a linear combination of $u$ and $v$ and the OTHER IS NOT.

   (a) Is the information given consistent with $w$ being a linear combination of $u$ and $v$?
   If not, say '$w$ is NOT a linear combination of $u$ and $v$.' If so, find a possible linear combination of $u$ and $v$ that could give $w$. Explain your conclusions!

   (b) Is the information given consistent with $z$ being a linear combination of $u$ and $v$?
   If not, say '$z$ is NOT a linear combination of $u$ and $v$.' If so, find a possible linear combination of $u$ and $v$ that could give $z$. Explain your conclusions!

19. Find all solutions of the system:

$$\left\{\begin{array}{rrrrrrl} & -\ b & +\ 2c & -\ d & +\ e & -\ 2f & = \ \ 4 \\ a & +\ 2b & & +\ d & -\ 3e & +\ f & = -3 \\ a & +\ 3b & -\ c & -\ 2d & & +\ f & = -3 \\ a & +\ 8b & +\ 2c & -\ 6d & +\ 2e & +\ 2f & = \ \ 1 \\ 2a & & -\ 3c & -\ d & -\ e & -\ f & = -6 \end{array}\right.$$

20. Find all solutions of the system:

$$\begin{cases} a - \phantom{2}b + 3c - 2d \phantom{{}+e} + \phantom{2}f = -2 \\ \phantom{a+{}} 2b - \phantom{3}c - \phantom{2}d + \phantom{2}e - 2f = \phantom{-}2 \\ a \phantom{{}+2b} + 2c + \phantom{2}d - 3e + \phantom{2}f = -3 \\ a + 2b + 8c - 6d + 2e + 2f = -1 \\ 2a - 3b \phantom{{}+8c} - \phantom{2}d - \phantom{2}e - \phantom{2}f = -3 \end{cases}$$

21. Let the basis $\mathcal{B}$ for $\mathbf{C}^4$ consist of $u = (1, -1, 1, 0)$, $v = (1, -2, 0, 1)$, $w = (2, 0, 1, 1)$, and $x = (0, 1, 1, 2)$. Let $T$ be the linear transformation defined on $\mathbf{C}^4$ by

$$Tu = 2u + v, \quad Tv = 2v + w, \quad Tw = 2w + x, \quad \text{and} \quad Tx = 2x + u$$

(a) Find the matrix for $T$ with respect to the basis $\mathcal{B}$.

(b) Let $y = 3u - v + x$. Find $T(y)$ and express your answer in terms of $u$, $v$, $w$, and $x$.

(c) Find the matrix for $T$ with respect to the standard basis, $\{e_1, e_2, e_3, e_4\}$.

(d) Find $T(z)$ where $z = (1, 2, 3, 1)$.

22. Find an orthonormal basis for the subspace of $\mathbf{R}^5$ spanned by $u_1 = (1, 1, 1, 0, 1)$, $u_2 = (1, 0, 2, 1, -1)$, $u_3 = (2, 1, 0, -1, 1)$, and $u_4 = (2, 2, -1, -2, 3)$.

23. Let the basis $\mathcal{B}$ for $\mathbf{C}^4$ consist of $u = (-1, 1, 1, 0)$, $v = (-1, 2, 0, 1)$, $w = (-2, 0, 1, 1)$, and $x = (0, -1, 1, 2)$. Let $T$ be the linear transformation defined on $\mathbf{C}^4$ by

$$Tu = u + 2v, \quad Tv = v + 2w, \quad Tw = w + 2x, \quad \text{and} \quad Tx = x + 2u$$

(a) Find the matrix for $T$ with respect to the basis $\mathcal{B}$.

(b) Let $y = u - 3v + x$. Find $T(y)$ and express your answer in terms of $u$, $v$, $w$, and $x$.

(c) Find the matrix for $T$ with respect to the standard basis, $\{e_1, e_2, e_3, e_4\}$.

(d) Find $T(z)$ where $z = (2, -1, 3, 1)$.

24. Let $M$ be the subspace of $\mathbf{R}^5$ spanned by $u_1 = (1, 1, -1, -1, 0)$, $u_2 = (1, -1, -1, 0, 1)$, $u_3 = (-1, -1, 0, 1, 1)$, and $u_4 = (1, -1, -2, 0, 2)$. Find an orthonormal basis for $M^\perp$.

25. Let $\mathcal{W}$ be the subspace of $\mathbf{R}^5$ spanned by $w_1 = (1, 1, 0, -1, 1)$, $w_2 = (0, 2, -1, 1, 1)$, and $w_3 = (-1, 1, 1, -2, 1)$. Find $x$ in $\mathcal{W}$ and $y$ in $\mathcal{W}^\perp$ so that $x + y = (1, 1, -1, -1, 1)$.

26. Let $M$ be the subspace of $\mathbf{R}^5$ spanned by $u_1 = (1, -1, -1, 0, 1)$, $u_2 = (1, -1, 0, 1, -1)$, $u_3 = (-1, 0, 1, 1, -1)$, and $u_4 = (1, -2, 0, 2, -1)$. Find an orthonormal basis for $M^\perp$.

27. Let $E = \begin{pmatrix} .6 & .5 & .4 \\ .1 & 0 & .2 \\ .2 & .4 & .3 \end{pmatrix}$ State a theorem using norms of matrices that guarantees $(I - E)^{-1}$ exists and verify the hypothesis of that theorem.

28. Let $B = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 4 & -3 \\ 2 & 0 & 1 \end{pmatrix}$

(a) Find $\|B\|_1$.

(b) State a theorem that implies that $8I - B$ is invertible (verify the hypotheses!) and use the first three terms of an appropriate series to find an approximation for $(8I - B)^{-1}$.

29. A simple model for the drag of a moving body is $D = av + bv^2$ where $D$ is the force due to friction, $v$ is the velocity of the object, and $a$ and $b$ are constants that depend on the material the object is made from.

Find the best (least squares) estimates for $a$ and $b$ in this model based on the data below:

Drag Data

| Force | Velocity |
|---|---|
| 2 | 1 |
| 7 | 2 |
| 18 | 5 |
| 45 | 8 |

30. Far Out (FO) and Way Back (WB) are enemy countries on the planet Utopia, which, surprisingly, is a completely flat, rather narrow planet! The army of Far Out (FO) has fired a rocket toward Way Back (WB) that is following a parabolic trajectory, $y = \alpha x^2 + \beta x + \gamma$, where $y$ is the height of the rocket above the ground and $x$ is the position on the planet Utopia, where $x = 0$ is the capital of WB, $x = -2$ is the second largest city of WB, and $x = 10$ is the capital of FO. Engineers in WB have observed the rocket's launch and have measured its position four times.

Rocket Trajectory Data

| position (x) | height (y) |
|---|---|
| 10 | 18 |
| 9 | 30 |
| 8 | 50 |
| 7 | 55 |

(a) Estimate (using the 'least squares criterion') the coefficients in the formula for the parabola that describes the trajectory of the rocket, $y = \alpha x^2 + \beta x + \gamma$.

(b) Assuming that your estimates for the coefficients in the formula for the parabola that describes the trajectory of the rocket are correct, from what position was the rocket launched, and at what position will the rocket hit the ground?

31. A political consultant is helping potential candidate Edwards decide whether to enter the primary for governor. The consultant believes there are three important qualities by which the voters rank the candidates: charisma ($c$), sincerity ($s$), and approachability ($a$). There are already four announced candidates, Adams, Baker, Chase, and Davis, and there is polling data to help determine their electability ($E$). The consultant believes there is an equation $E = \alpha c + \beta s + \gamma a$ that predicts electability.

Electability Data

| | charisma (c) | sincerity (s) | approach. (a) | Electability (E) |
|---|---|---|---|---|
| Adams | .2 | .7 | .4 | 37 |
| Baker | .4 | .5 | .3 | 29 |
| Chase | .5 | .2 | .6 | 46 |
| Davis | .3 | .6 | .4 | 32 |
| Edwards | .6 | .4 | .4 | ? |

(a) Estimate (using the 'least squares criterion') the coefficients $\alpha$, $\beta$, and $\gamma$ in the formula $E = \alpha c + \beta s + \gamma a$ that the consultant believes describes electability based on the polling data given above for Adams, Baker, Chase, and Davis.

(b) By comparing Edward's qualities to the polling data for the present candidates, the consultant has projected polling data for Edwards and this is entered in the table also. Use your estimates for the coefficients in the formula to predict an electability for Edwards. If Edwards wants to be assured of beating at least two of the candidates, should the consultant advise him to run?