Questions from Old Final Exams for Math 51100

1. Solve the following system of equations. If there are no solutions, say so; if the solution is unique, say so; if there are infinitely many solutions, find the general solution and give two solutions explicitly.

$$\begin{cases} a - b + c + d + e = 0 \\ a - 3b + 5c + 7d + 9e = -4 \\ a - 2b + 2c + 3d + 3e = 1 \\ 4a - 5b + 5c + 6d + 6e = 1 \end{cases}$$

2. The matrix A is a 5×6 real matrix, X represents a vector in $\mathbb{R}^6$ and b is a vector in $\mathbb{R}^5$.
Consider the following two systems of equations:

$$(H) \;\; AX = 0 \qquad \text{and} \qquad (N) \;\; AX = b$$

The vectors $(0, -1, 1, 2, 1, 0)$ and $(0, 2, -2, -4, -2, 0)$ are solutions of system (H).
The vectors $(2, 0, 1, 1, 1, 1)$ and $(-2, 1, 1, 0, 2, 1)$ are solutions of system (N).

(a) Find two linearly independent solutions of (H) different from those given above.

(b) Find two solutions of (N) different from those given above.

3. The matrix E is a 5×6 real matrix, X represents a vector in $\mathbb{R}^6$ and d is a vector in $\mathbb{R}^5$.
Consider the following two systems of equations:

$$(H) \;\; EX = 0 \qquad \text{and} \qquad (N) \;\; EX = d$$

The vectors $(0, -2, -1, 3, 2, 0)$ and $(0, 4, 2, -6, -4, 0)$ are solutions of system (H).
The vectors $(1, 1, 3, 0, 1, 1)$ and $(-2, 1, 1, 0, 2, 1)$ are solutions of system (N).

(a) Find two linearly independent solutions of (H) different from those given above.

(b) Find two solutions of (N) different from those given above.

4. Let $B = \begin{pmatrix} 2 & 2 & -1 & -3 \\ 2 & 2 & -3 & -1 \\ -1 & -3 & 2 & 2 \\ -3 & -1 & 2 & 2 \end{pmatrix}$

(a) Give facts from linear algebra theory that explain why $\mathcal{R}(B)$, the range of B, and $\mathcal{N}(B)$, the null space of B, are orthogonal complements, that is, explain why $\mathcal{N}(B) = \mathcal{R}(B)^{\perp}$.

(b) For $z = \begin{pmatrix} 2 \\ -1 \\ 4 \\ 3 \end{pmatrix}$ write z as $z = x + y$ where x is in $\mathcal{R}(B)$ and y is in $\mathcal{N}(B)$.

5. Let $F = \begin{pmatrix} 2 & 1 & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 2 & 2 & 1 \\ 2 & 3 & 1 & 2 \end{pmatrix}$

(a) Give facts from linear algebra theory that explain why $\mathcal{R}(F)$, the range of F, and $\mathcal{N}(F)$, the null space of F, are orthogonal complements, that is, explain why $\mathcal{N}(F) = \mathcal{R}(F)^{\perp}$.

(b) For $z = \begin{pmatrix} -2 \\ 2 \\ 1 \\ 4 \end{pmatrix}$ write z as $z = x + y$ where x is in $\mathcal{R}(F)$ and y is in $\mathcal{N}(F)$.

6. For each of the situations (a)–(f) below, decide which of the statements in the box can correctly complete the sentence. *Include all correct responses.*

(a) If A is an 8×13 matrix whose rank is 6, then ________

(b) If A is an 8×13 matrix whose rank is 8, then ________

(c) If A is an 8×13 matrix whose rank is 10, then ________

(d) If A is a 13×7 matrix whose rank is 9, then ________

(e) If A is a 13×7 matrix whose rank is 7, then ________

(f) If A is a 13×7 matrix whose rank is 5, then ________

> (i) $AX = b$ is solvable for every vector b.
> (ii) there are some vectors b for which $AX = b$ is not solvable.
> (iii) for some vectors b, the system $AX = b$ has exactly one solution.
> (iv) for some vectors b, the system $AX = b$ has infinitely many solutions.
> (v) the given information is contradictory, no such system is possible.

7. The matrix C is a 7×9 matrix and the dimension of $\mathcal{R}(C)$, the range of C, is 4.

(a) What is the dimension of $\mathcal{N}(C)$, the nullspace of C?________

(b) What is the dimension of $\mathcal{R}(C')$, the range of C'?________

(c) What is the dimension of $\mathcal{N}(C')$, the nullspace of C'?________

(d) What is the dimension of $\mathcal{N}(C')^{\perp}$, the orthogonal complement of $\mathcal{N}(C')$?________

8. M is the subspace spanned by $u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ $u_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ $u_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ and $u_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

(a) Find an orthonormal basis for M.

(b) Find vectors u and v such that u is in M, v is in $M^{\perp}$, and $u + v = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \\ 2 \end{pmatrix}$

9. The following system is inconsistent.

$$\begin{cases} x + y - z = 11 \\ x - 2y - z = -0.5 \\ -2x + y + 2z = -1 \\ -3x + 2y + z = -2 \end{cases}$$

(a) Find the least squares solution of this system.

(b) Let C be the coefficient matrix for this system, that is, $C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & -1 \\ -2 & 1 & 2 \\ -3 & 2 & 1 \end{pmatrix}$

Letting $b = \begin{pmatrix} 11 \\ -0.5 \\ -1 \\ -2 \end{pmatrix}$, what vector in $\mathcal{R}(C)$, the range of C, is closest to the vector b?

What is the distance from b to $\mathcal{R}(C)$, the range of C?

10. The matrix J is a 4×4 real matrix whose eigenvalues are 2, 3, and 1:

$\begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 2$;

$\begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 3$;

and $\begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ are a basis for the eigenspace corresponding to $\lambda = 1$.

(a) Is J diagonalizable? YES NO Cannot be determined from the given information

(b) Is $J' = J$? YES NO Cannot be determined from the given information

(c) Is J positive definite? YES NO Cannot be determined from the given information

(d) Find the two eigenvalues of $J^3 - 2J^2 - J + 5I$ and find bases for the corresponding eigenspaces.

(e) Find Jw where $w = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -4 \end{pmatrix}$

11. F is a 3×4 matrix that satisfies

$$F\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad F\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad \text{and} \quad F\begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

(a) Find a vector X such that $FX = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(b) Find two vectors $Y_1 \neq Y_2$ such that $FY_1 = FY_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

(c) Is there a vector Z, with Z DIFFERENT FROM $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, for which $FZ = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$?

If not, explain why not. If so, find another such Z.

12. Let H be an 5×5 matrix whose (only) eigenvalues are $\lambda_1 = -3$, $\lambda_2 = -2$, $\lambda_3 = 3$, and $\lambda_4 = 4$

(a) What are the eigenvalues of $J = H^2 + 5H + 2I$.

(b) If v is an eigenvector for H with eigenvalue 3, what is Jv, where $J = H^2 + 5H + 2I$ as above?

(c) Explain, that is, using a theorem, how you know that H is invertible.

(d) What are the eigenvalues of H^{-1}?

13. (a) What property do the vectors $p = (1, -2, -2)$ and $q = (3, 0, 0)$ have that make it possible for there to be a unitary matrix U so that $Up = q$? Find a unitary matrix U so that $Up = q$.

(b) Supposing you have found matrix U above, explain how to find a unitary matrix V so that $Vq = p$.

14. The matrix A is a *square*, $n \times n$ matrix and b is a vector in $\mathbb{R}^n$.

In each of the following, a condition is given and then a statement. When the given condition is true, decide if the statement is *always true* or *always false* or *sometimes true, sometimes false*, and circle the appropriate answer.

(a) **Condition:** $\det(A) = 0$.
Statement: The equation $AX = b$ has no solutions.

always true | always false | sometimes true, sometimes false

(b) **Condition:** The vectors $w_1, w_2, \cdots, w_j$ are linearly independent.
Statement: The vectors $Aw_1, Aw_2, \cdots, Aw_j$ are linearly independent.

always true | always false | sometimes true, sometimes false

(c) **Condition:** The vectors $Aw_1, Aw_2, \cdots, Aw_j$ are linearly dependent.
Statement: The vectors $w_1, w_2, \cdots, w_j$ are linearly dependent.

always true | always false | sometimes true, sometimes false

(d) **Condition:** A is a 3×3 Hermitian matrix with characteristic polynomial
$$\lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda - 1)^2.$$
Statement: There is a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

always true | always false | sometimes true, sometimes false

(e) **Condition:** The matrix A is invertible.
Statement: The columns of the matrix A are an orthonormal set of vectors..

always true | always false | sometimes true, sometimes false

15. The matrix B is a *square*, $n \times n$ matrix and c is a vector in $\mathbb{R}^n$.

In each of the following, a condition is given and then a statement. When the given condition is true, decide if the statement is *always true* or *always false* or *sometimes true, sometimes false*, and circle the appropriate answer.

(a) **Condition:** The vectors $w_1, w_2, \cdots, w_j$ are linearly dependent.
Statement: The vectors $Bw_1, Bw_2, \cdots, Bw_j$ are linearly independent.

always true | always false | sometimes true, sometimes false

(b) **Condition:** The equation $BX = c$ has infinitely many solutions.
Statement: $\det(B) = 0$.

always true | always false | sometimes true, sometimes false

(c) **Condition:** The columns of the matrix B are an orthonormal set of vectors.
Statement: The matrix B is invertible.

always true | always false | sometimes true, sometimes false

(d) **Condition:** B is a 3×3 matrix with characteristic polynomial
$$\lambda^3 - \lambda = \lambda(\lambda - 1)(\lambda + 1).$$
Statement: There is a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

always true | always false | sometimes true, sometimes false

(e) **Condition:** B is an $n \times n$ Hermitian matrix that is not invertible.
Statement: There is a basis for $\mathbb{R}^n$ consisting of eigenvectors of B.

always true | always false | sometimes true, sometimes false

16. Let $L = \begin{pmatrix} 4 & 10 & 0 & -10 \\ -2 & 8 & -1 & -3 \\ 0 & -5 & 4 & 5 \\ -2 & 9 & -1 & -4 \end{pmatrix}$

   (a) Find the three eigenvalues of L
   (b) Find a basis for each of the eigenspaces for the eigenvalues (identifying which) in part (a).
   (c) Find a basis of $\mathbb{R}^4$ consisting of eigenvectors of L.
   (d) Find three eigenvectors, u, v, and w of L such that $u + v + w = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

17. Let $G = \begin{pmatrix} -2 & -4 & -8 & 12 \\ 9 & 9 & 10 & -9 \\ 9 & 3 & 16 & -9 \\ 3 & 1 & 2 & 7 \end{pmatrix}$

   (a) Find the three eigenvalues of G
   (b) Find a basis for each of the eigenspaces for the eigenvalues (identifying which) in part (a).
   (c) Find a basis of $\mathbb{R}^4$ consisting of eigenvectors of G.
   (d) Find three eigenvectors, u, v, and w of G such that $u + v + w = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$.

18. The matrix $K = \begin{pmatrix} 2 & 6 & -3 & 1 \\ 0 & -1 & 1 & 0 \\ 3 & 7 & -4 & 1 \\ 4 & 8 & -7 & 3 \end{pmatrix}$ has two positive and two negative eigenvalues.

   Let M be the subspace spanned by the eigenvectors corresponding to the negative eigenvalues of K. (The subspace M is called the *stable manifold* of K.)

   (a) Find the matrix for the orthogonal projection of $\mathbf{C}^4$ onto M.
   (b) Find the point of the stable manifold M that is closest to $(1, -1, 1, 0)$.

19. The matrix $K = \begin{pmatrix} 7 & 3 & 2 & 1 \\ 4 & -1 & 1 & 1 \\ 3 & 0 & 2 & -1 \\ -3 & 2 & -2 & 0 \end{pmatrix}$ has two positive and two negative eigenvalues.

   Let M be the subspace spanned by the eigenvectors corresponding to the negative eigenvalues of K. (The subspace M is called the *stable manifold* of K.)

   (a) Find the matrix for the orthogonal projection of $\mathbf{C}^4$ onto M.
   (b) Find the point of the stable manifold M that is closest to $(1, -1, 1, 0)$.

20. (a) Let $B = \begin{pmatrix} 5 & 0 & -2 & -2 \\ -3 & -2 & 8 & 18 \\ 3 & 3 & -3 & -12 \\ -2 & -3 & 5 & 14 \end{pmatrix}$
   Show that $x = (1, 1, 1, 0)$ and $y = (2, -3, 3, -2)$ are eigenvectors of B, but $z = (1, 1, 0, 0)$ is not.
   (b) Find the eigenvector of B with eigenvalue 3 that is closest to z.

21. Let A be an $n \times n$ matrix such that $A' = A = A^{-1}$. Let $P = \dfrac{1}{2}(I - A)$.
Prove that $P = P'$ and that $P^2 = P$.

22. Let D be an $n \times n$ matrix with $D' = D$ and $\text{rank}(D) = n - k$. Suppose $v_1, v_2, \cdots, v_k$ are linearly independent vectors such that $Dv_j = 0$ and suppose w is a vector such that $\langle v_j, w \rangle = 0$ for $j = 1, 2, \cdots, k$. Prove that there is a vector u so that $Du = w$.

23. (a) Suppose R and S are $n \times n$ matrices such that $RS = SR$. Let u be an eigenvector for R with eigenvalue α. Prove that either Su is zero or Su is also an eigenvector for R with eigenvalue α.
(b) Suppose R, S, u, and α are as in part (a) and suppose, in addition, that the eigenspace of R corresponding to α is one–dimensional. Prove that in this case, u is an eigenvector for S also.

24. (a) Suppose u is an eigenvector for A' and v is orthogonal to u. Show that Av is also orthogonal to u.
(b) Use part (a) (whether you proved it or not) to show that if A is a 2×2 Hermitian matrix, and u is an eigenvector of A, then any non-zero vector v that is orthogonal to u is also an eigenvector of A.