This is an extension of the Midterm Test 1. If you wish to submit corrections to your answers on the test questions, you can earn up to half the points that were deducted on the test paper for each question. To do this, you should submit your original, graded, test paper with your old work unchanged and submit your corrections on a new sheet of paper with this homework.

1. (See Test question 2.)
For each positive integer n, let

$$f_n(x) = \frac{x}{x+n}$$

and let $g(x) = \lim_{n\to\infty} f_n(x)$ for $x \geq 0$.

(a) Does the sequence (f_n) converge uniformly to g on $[0, 2]$? Justify your answer.

(b) Does the sequence (f_n) converge uniformly to g on $[0, \infty)$? Justify your answer.

2. (See Test question 3 and Homework 1.)

(a) Prove that if $\sum_{k=1}^{\infty} a_k = L$, that is, the series is convergent and the sum of the series is L, then the series is Cesáro summable to L.

(b) Show that the Alternating Harmonic Series, $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$, is Cesáro summable and find the number L.

(c) Is the Harmonic Series, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, Cesáro summable? Justify your answer.

3. (See Test question 4.)

(a) Give an example of a sequence of contractions f_n, each mapping $[0, 1]$ into itself, such that the sequence (f_n) converges uniformly to g on $[0, 1]$ but g is not a contraction.

(b) Let (f_n) be a sequence of continuous functions that map the closed interval $[\alpha, \beta]$ into itself such that each f_n has a fixed point in $[\alpha, \beta]$. Prove that if the sequence (f_n) converges uniformly to the function g, then g maps the interval $[\alpha, \beta]$ into itself and g has a fixed point also.

(c) Let (f_n) be a sequence of contractions that map the interval $[0, \infty)$ into itself and suppose that the sequence (f_n) converges uniformly to the function g. Either prove that g must have a fixed point in $[0, \infty)$ or give an example of a sequence of contractions (f_n) and g satisfying the conditions above for which g does not have a fixed point in $[0, \infty)$.

4. Use Abel's Theorem and an appropriate power series to find the sum of the series

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} - \frac{1}{6} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} - \frac{1}{8} + \cdots$$

(You may assume without proof that this series converges.)