Application of Linear Algebra to Differential Equations

Segment 1: Introduction

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OUTLINE

References: Section 8.3, Section 10.2

Problems: For Discussion May 1: page 328: 1, 2, 3, 4, 5 page 392: 1, 2, 4

Differential Equation: an equation involving an unknown function,

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Solving a differential equation means finding all functions that satisfy the equation

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Solution: integrate twice to get $x(t) = -\frac{g}{2}t^2 + v_0t + x_0$

where v_0 is the initial velocity and x_0 is the initial height.

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Solution: Rewrite as $\frac{1}{y}\frac{dy}{dt} = a$ to get $\ln(y) = at + c$ or $y(t) = Ce^{at}$

where $C = e^c$

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Second derivative of unknown function y occurs, but no higher derivative,

so the equation is said to be a *second order* equation.

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collection of equations with several unknown functions and their derivatives in which each equation is linear and all the coefficients are constants.

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For example, let's look at the system
$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = 2y_1 + 4y_2 \end{cases}$$

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That is, the system becomes $\quad Y' = AY \quad$ where $\quad A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

$Y' = AY$ is a linear system so the following is not surprising:

Theorem (*Principle of Superposition*):

If A is an $n \times n$ matrix with U and V solutions of the system $Y' = AY$, then for any numbers α, β, the function $W = \alpha U + \beta V$ is also a solution.

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Proof: If $W(t) = \alpha U(t) + \beta V(t)$, then $W'(t) = \alpha U'(t) + \beta V'(t)$.

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so $\alpha U' = \alpha AU = A(\alpha U)$ and $\beta V' = \beta AV = A(\beta V)$. This means

$$W' = \alpha U' + \beta V' = A(\alpha U) + A(\beta V) = A(\alpha U + \beta V) = AW$$

which is the conclusion. ∎

This is the end of the First Segment.

In the next segment, we will investigate the matrix exponential so that we can deal with the equation $Y' = AY$ in a way analogous to the equation $y' = ay$.