February 21 to 26 The following problems will not be collected but might be helpful in preparing for the midterm and might be discussed in the Linear Algebra Seminar next Monday.

44. (a) Let f be the polynomial $f(x) = x^3 - 4x^2 + 3x - 5$. Let B be a 3×3 invertible matrix that satisfies $f(B) = 0$. Find a polynomial g so that $B^{-1} = g(B)$.
(Hint: rewrite the equation $f(B) = 0$ in such a way as to get I alone on the right side of the equation.)

(b) In Exercise 29, it was shown that every 3×3 matrix, A, satisfies a polynomial equation $p(A) = 0$ for some non-zero polynomial. The same can be done for $n \times n$ matrices: If A is an $n \times n$ matrix, there is a non-zero polynomial p for which $p(A) = 0$. Assuming that result has been proved, show that for every invertible $n \times n$ matrix A, there is a polynomial q so that $A^{-1} = q(A)$.

45. Let n be a positive integer and F a field. Suppose A is an $n \times n$ matrix over F and P is an invertible $n \times n$ matrix over F. Prove: If f is any polynomial, then

$$f(P^{-1}AP) = P^{-1}f(A)P$$

46. Let $\mathbb{Q}$ the field of rational numbers. Determine which of the following are ideals in $\mathbb{Q}[x]$. If the set is an ideal, find a monic generator. If it is not an ideal, explain why it is not.

(a) All polynomials with even degree.

(b) All polynomials f with degree$(f) \geq 5$.

(c) All polynomials f for which $f(2) = f(4) = 0$.

(d) All polynomials f for which $f(2) - f(4) = 0$.

(e) The range of the linear transformation $T(f) = (5x^2 + 2)f$, for f in $\mathbb{Q}[x]$.

47. Let F be a subfield of the complex numbers.

(a) Let A be an $n \times n$ matrix over F.
Prove: the set of polynomials in $F[x]$ for which $f(A) = 0$ is an ideal.

(b) Find the monic generator of the ideal of polynomials in $F[x]$ for which $f(A) = 0$ when

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$$

48. Let F be a field and let $F[x]$ be the algebra of polynomials over F.

(a) Prove: If $a \neq 0$ and b are elements of F, the polynomials 1, $ax + b$, $(ax + b)^2$, $(ax + b)^3$, $\cdots$, form a basis for $F[x]$.

(b) More generally, show that if h is a polynomial in F of degree at least 1 then the mapping $T(f) = f(h)$ is a linear transformation of $F[x]$ into itself.

(c) Show that the transformation T in part (b) is an isomorphism of $F[x]$ onto $F[x]$ if and only if h has degree 1.

49. Let F be a field and let $F[x]$ be the algebra of polynomials over F.

(a) Prove that the intersection of any number of ideals in $F[x]$ is also an ideal in $F[x]$.

(b) Let f_1, f_2, $\cdots$, f_k be polynomials in $F[x]$ and let J be the ideal generated by $\{f_1, f_2, \cdots, f_k\}$. Show that J is the intersection of all of the ideals in $F[x]$ that contain all of the f_j for $j = 1, \cdots, k$

50. An $n \times n$ matrix $T = (t_{ij})$ is said to be a *Toeplitz matrix* if $t_{ij} = t_{i+1,j+1}$ for $1 \leq i, j < n$.

(a) Prove: If S and T are a lower triangular $n \times n$ Toeplitz matrices, then ST is a lower triangular Toeplitz matrix also.

(b) Give an example to show that if S and T are both $n \times n$ Toeplitz matrices, then it is not necessarily the case that ST is a Toeplitz matrix.

(c) Prove: If $T = (t_{ij})$ is a lower triangular $n \times n$ Toeplitz matrix with $t_{11} \neq 0$, then T is invertible and T^{-1} is also a Toeplitz matrix.

(d) Let T be the 4×4 Toeplitz matrix with $t_{1,1} = 1$, $t_{2,1} = -2$, and $t_{3,1} = 1$ with $t_{4,1} = t_{1,2} = t_{1,3} = t_{1,4} = 0$. Find T^{-1}.

(e) Let T be the $n \times n$ Toeplitz matrix with $t_{1,1} = 1$, $t_{2,1} = -2$, and $t_{3,1} = 1$ and $t_{i,j} = 0$ for $i - j \neq 0, 1,$ or 2. Make a conjecture for T^{-1}. Can you prove your conjecture?

Comments

- The IUPUI Math Sciences Department has copies of old qualifying exams on the department website. The URL for the Math 55400 exams is
  `math.iupui.edu/sites/default/files/55400_quals_2007_thru_2019.pdf`
  The exams are set by the instructor who last taught the course. It follows that questions on midterm tests and final exams in Math 55400, will have questions that are like those on the qualifying exams. In the Mock Test 1 below, the numbers in parentheses are year 2 digits and month two digits for the qualifying exams those questions were on: August of '07 and '11.
- There are 5 or 6 questions on most of the qualifying exams (2 hours), so there will probably be 5 questions on the midterm and 5 or 6 questions on the final exam.
- Do the "easy" (for you) questions first and don't spend too much time on any one problem.
- Full credit for exam questions is about proving your assertions, not just outlining the steps.

Mock Test 1

- Proofs should quote results from the class to justify assertions, as in 'By the rank-nullity theorem, we can see $\cdots$'.
- If you need to use a property of a vector space or transformation or use a theorem in your answer, explain what property or theorem you are using and indicate specifically why the hypotheses of the theorem are satisfied.
- Explain your answers for each question in such a way that your reasoning can be followed!!

51. (0708) Let $\mathcal{V}$ be a finite dimensional vector space over the field $\mathbb{F}$. Show that the vectors v_1, v_2, $\cdots$, v_k are a basis for $\mathcal{V}$ if and only if, for any non-zero linear functional f in the dual of $\mathcal{V}$, there is v_j with $1 \leq j \leq k$ for which $f(v_j) \neq 0$.

52. (0708) Suppose A and B are complex $n \times n$ matrices for which $AB = 0$. Prove that $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B) \leq n$.

53. (1108) Let S be a linear transformation on the finite dimensional vector space $\mathcal{V}$ over the field $\mathbb{F}$ that satisfies $S^m = S$ for some positive integer $m > 1$.

(a) Letting $\mathcal{N}(S)$ and $\mathcal{R}(S)$ be the nullspace and range of S, prove that $\mathcal{N}(S) \cap \mathcal{R}(S) = (0)$

(b) Prove that $\mathcal{N}(S) = \mathcal{N}(S^k)$ for every positive integer k.

54. An $n \times n$ matrix N over a field $\mathbb{F}$ is called *a nilpotent matrix of order k* if $N^k = 0$, but $N^{k-1} \neq 0$. For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

are nilpotent matrices of order 3 and 2 respectively. Let N be an $n \times n$ matrix that is nilpotent of order 4. Find the inverse of $I + 2N + N^2 - 3N^3$.

55. We have seen that for $\mathcal{V}$ a finite dimensional vector space over the field $\mathbb{F}$, T a linear transformation on $\mathcal{V}$, and v a vector in $\mathcal{V}$, then $J = \{p \in \mathbb{F}[x] : p(T)v = 0\}$ is an ideal in the ring of polynomial over $\mathbb{F}$.

(a) Suppose $\mathcal{V}$ is a 3-dimensional vector space over the field $\mathbb{F}$, v is a vector in $\mathcal{V}$, and T is a linear transformation on $\mathcal{V}$. Prove that there is a polynomial, p, of degree 3 for which $p(T)v = 0$.

(b) Let $D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and let $v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. Show that there is no (monic) polynomial of degree 2 in the ideal $K = \{q \in \mathbb{R}[x] : q(D)v = 0\}$.

(c) Find a monic polynomial of degree 3 in the ideal K in part (b) and explain why this means that the polynomial you just found is the monic generator of K.

(d) Let $C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$. Choose two linearly independent vectors u and w in $\mathbb{R}^3$.

Let $K_u = \{q \in \mathbb{R}[x] : q(C)u = 0\}$ and $K_w = \{q \in \mathbb{R}[x] : q(C)w = 0\}$ be the ideals in $\mathbb{R}[x]$ associated with C and u and w. For each of the ideals K_u and K_w, find a monic polynomial of degree 3 in the ideal.