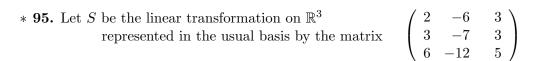
94. Let T be the linear transformation on \mathbb{C}^3 whose matrix with respect to the usual basis is

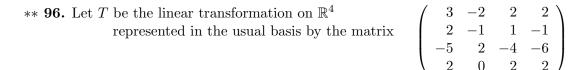
$$\left(\begin{array}{rrrr} 1 & i & 0 \\ -1 & 2 & -i \\ 0 & 1 & 1 \end{array}\right)$$

- (a) Find the *T*-annihilator of (1, 0, 0).
- (b) Find the *T*-annihilator of (1, 0, i).



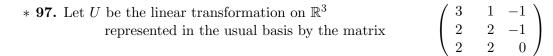
If p is the minimal polynomial for a matrix, Theorem 12 from class and the text uses the factorization $p = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where k and r_1, \cdots, r_k are all positive integers.

- (a) Express the minimal polynomial for S as $p = p_1^{r_1} p_2^{r_2}$ where p_1 and p_2 are monic, irreducible polynomials over \mathbb{R} .
- (b) For both j = 1 and j = 2, find a basis \mathcal{B}_j for W_j , the null space of $p_j(S)$.
- (c) Find the matrices for S_1 and S_2 , the restrictions of S to W_1 and W_2 , with respect to these bases, and also find the matrix for S with respect to the basis $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2\}$.



Note that \mathbb{R}^4 is a real vector space, not a complex vector space.

- (a) Find the minimal polynomial of T.
- (b) Find the characteristic polynomial of T.
- (c) Factor each of these polynomials as a product of monic irreducible polynomials over \mathbb{R} .
- (d) Using Theorem 12 and your answer to (c), identify k, r_1, \dots, r_k and polynomials p_1, \dots, p_k as in the theorem.
- (e) Using the notation of Theorem 12, find a basis for each of the subspaces, W_1, \dots, W_k .
- (f) For each j, with $1 \le j \le k$, using the notation of Theorem 12, find the matrix for T_1 , \dots , T_k , each with respect to the appropriate basis found above.
- (g) Find the matrix for T with respect to the basis for \mathbb{R}^4 that comes from combining the bases for W_1, \dots, W_k .



Show that there are a diagonalizable operator D and a nilpotent operator N on \mathbb{R}^3 so that U = D + N and DN = ND. Find the matrices for D and N in the usual basis for \mathbb{R}^3 .

- **98.** Let T be a linear transformation on the finite dimensional vector space \mathcal{V} .
 - (a) Prove that if T^2 has a cyclic vector, then T has a cyclic vector.
 - (b) Is the converse true? Either give a proof or a counterexample to show that your answer is correct.
- * 99. Let N be a nilpotent linear transformation on the n-dimensional vector space \mathcal{V} .
 - (a) Prove: N has a cyclic vector if and only if $N^{n-1} \neq 0$.
 - (b) If v is a vector in \mathcal{V} for which $N^{n-1}v \neq 0$, what is the matrix for N with respect to the basis $v, Nv, \dots, N^{n-1}v$.
- 100. Prove that if A and B are 3×3 matrices over the field F, then A and B are similar if and only if they have the same minimal polynomials and the same characteristic polynomials. Give an example that shows this is not a theorem for 4×4 matrices.
- * 101. Let C be a linear transformation on a finite dimensional vector space \mathcal{V} .
 - (a) Prove: If C does not have a cyclic vector, there is an transformation G that commutes with C, but G is not a polynomial in C.
 - (b) Prove: If C has a cyclic vector, every transformation that commutes with C is a polynomial in C.

In other words, C has a cyclic vector if and only if every transformation that commutes with C is a polynomial in C.

102. (a) Let A be a linear transformation on the vector space V and let v₁, v₂, ..., v_k be vectors in V.
Prove: If the set {Av₁, Av₂, ..., Av_k} is a linearly independent set,

then the set $\{v_1, v_2, \cdots, v_k\}$ is also linearly independent.

(b) Show that the converse of the statement in part (a) is false: that is, find a linear transformation T on a vector space \mathcal{V} and a set $\{v_1, v_2, \dots, v_k\}$ of vectors in \mathcal{V} that are linearly independent, but the set $\{T(v_1), T(v_2), \dots, T(v_k)\}$ is linearly dependent.