February 13

Definition: Let V be a vector space over the field F. A *linear functional on* V is a linear transformation from V into F.

Definition: Let A and B be two $n \times n$ matrices with entries in a field F. We say the *matrices* A *and* B *are similar* if there is an $n \times n$ invertible matrix S so that $B = S^{-1}AS$.

Definition: Let A be an $n \times n$ matrix with entries in a field F; as usual, let a_{jk} denote the entry of A in the j^{th} row and k^{th} column. The *trace of the matrix* A, written $\mathrm{tr}(A)$, is the function defined on the vector space of $n \times n$ matrices into the field F by $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{j=1}^{n} a_{jj}$.

$*$ **33.** Let V be the vector space of polynomial functions of $\mathbb{R}$ into $\mathbb{R}$ of degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2 \qquad \text{for} \quad c_0, c_1, \text{ and } c_2 \in \mathbb{R}$$

Let a and b be two real numbers and let J be the function on V into $\mathbb{R}$ defined by

$$J(p) = \int_a^b p(x)\,dx$$

Prove that J is a linear functional on V.

$*$ **34.** Show that the trace function is a linear functional on the vector space of $n \times n$ matrices with entries in the field F.

$**$ **35.** Let A, B, and C be $n \times n$ matrices over the field F.

(a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Prove that $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$.

(c) Give an example of matrices A, B, and C so that $\mathrm{tr}(ABC)$ is not the same as $\mathrm{tr}(BAC)$.

(d) Show that if A and B are similar matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

$*$ **36.** Show that $AB - BA = I$ is impossible for $n \times n$ matrices of real or complex numbers.