February 11

26. (a) Let $v_1 = (1, -1)$, $v_2 = (2, -1)$, and $v_3 = (-3, 2)$. Let $w_1 = (1, 0)$, $w_2 = (0, 1)$, and $w_3 = (1, 1)$. Is there a linear transformation T from $\mathbb{R}^2$ into $\mathbb{R}^2$ so that $T(v_j) = w_j$ for $j = 1, 2, 3$?

(b) Give necessary and sufficient conditions on u_1, u_2, and u_3 in $\mathbb{R}^2$ so that there is a linear transformation T from $\mathbb{R}^2$ into $\mathbb{R}^2$ for which $T(u_j) = w_j$ for $j = 1, 2, 3$?

$*$ **27.** Let V be an n-dimensional vector space over F and let T be a linear transformation for which the range of T and the null space of T are the same subspace.

(a) Prove that n is even.

(b) Give an example of a linear transformation S acting on $\mathbb{R}^4$ for which the range of S and the null space of S are spanned by $v_1 = (1, 0, 1, -1)$, $v_2 = (1, 1, -1, 2)$, and $v_3 = (3, 1, 1, 0)$.

$*$ **28.** Let V be a finite dimensional vector space over the field F and let T be a linear transformation for which the rank of T^2 and the rank of T are the same.
Prove that the intersection of the range of T and the null space of T is the zero subspace (0).

$*$ **29.** If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ where the a_j are real numbers, and A is an $n \times n$ real matrix, we define $p(A)$ to be $a_0 I + a_1 A + a_2 A^2 + \cdots + a_k A^k$.

(a) What is the dimension of the vector space of 3×3 real matrices?

(b) Prove: for any 3×3 real matrix A, there is a non-zero polynomial p so that $p(A) = 0$.

30. For n a positive integer, let T be a linear transformation of the vector space F^n into the space F^m and let A be the matrix for T with respect to the standard bases for F^n and F^m.
Let W be the subspace of F^m spanned by the columns of A.
State one or two relationships between W and T.

$*$ **31.** Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3) \tag{1}$$

(a) What is the matrix for T in the standard basis?

(b) Find the matrix for T relative to the basis

$$v_1 = (1, 0, 1) \quad v_2 = (-1, 2, 1) \quad v_3 = (2, 1, 1)$$

(c) Prove that T is invertible and find an expression for the transformation T^{-1} like the one in Equation (1).

$**$ **32.** Let V be an n-dimensional vector space over the field F and let $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ be an ordered basis for V.

(a) Let T be the (unique) linear transformation that satisfies $Tv_j = v_{j+1}$ for $j = 1, 2, \cdots, n-1$ and $Tv_n = 0$. Find the matrix for T with respect to basis $\mathcal{B}$ above.

(b) Prove that $T^n = 0$ but T^{n-1} is not the zero transformation.

(c) Let S be an operator on V for which $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is an ordered basis $\mathcal{B}'$ for V such that the matrix for S with respect to the basis $\mathcal{B}'$ is the same as the matrix from part (a) of this exercise.