Professor Carl Cowen **Math 55400** **Spring 19**

HOMEWORK PROBLEMS

Note: The dates in this problem list indicate the dates by which you should have answered/thought about these questions.

Note: $*$ indicates a problem that will be handed in on the listed date and graded. $**$ indicates a problem that will be handed in separately from the rest, graded, and has opportunity to be corrected (once).

January 9

$*$ **1.** We know that the set $\mathbb{Q}$ of rational numbers is a field. A common way to create new fields is to *adjoin* a root of a polynomial that has no roots in the original field. This exercise illustrates this process by creating the field F by adjoining to $\mathbb{Q}$ a root of the polynomial $x^2 - 2$ which has coefficients in $\mathbb{Q}$, but has no rational roots. That is, we create a symbol for a root of the polynomial $x^2 - 2$, and we create the field F by declaring that F is the smallest field that contains $\mathbb{Q}$ and element just named. As usual, the symbol for the root we are adjoining to create F is '$\sqrt{2}$'.

Prove that the set $F = \{p+q\sqrt{2}$: for rational numbers p and $q\}$ is a field by showing that addition and multiplication of any two elements or F are also in F, identifying the identities for addition and multiplication, and verifying the commutative and associative properties of these operations and the distributive property of multiplication over addition. Also, show that the additive and multiplicative inverses of any element of F are also in F. Finally, show that the element $0 + 1\sqrt{2}$ in F is a root of the polynomial $x^2 - 2$. Are there other roots of this polynomial in F?

The *Wikipedia* entry 'Finite Field', `http://en.wikipedia.org/wiki/Finite_field` gives some background that might be helpful in thinking about fields that are not subfields of the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$.

$**$ **2.** (a) Let $F_3 = \{0, 1, 2\}$ be the set of integers mod 3. Show that F_3 is a field with the usual operations of modular arithmetic by creating two 3×3 grids, one for addition and one for multiplication listing the results of sums of two elements of F_3 and the products of two elements of F_3. Point out how the tables identify the additive and multiplicative identities and inverses of the elements of F_3 and show that these operations are commutative. Show that the polynomial $x^2 - 2$ does not have a root in F_3.

(b) Create a field F_9 by adjoining a root, α, of the polynomial $x^2 - 2$ to F_3. That is, we know the polynomial $x^2 - 2$ does not have a root in F_3 but in building F_9, we declare F_9 has elements 0, 1, 2, and α, where $\alpha^2 - 2 = 0$, and we want to include other elements so that F_9 is a field such that there is no smaller subset of F_9 that is also a field. In this context, 'create' means making a list of the elements of F_9 (there should be nine elements) and creating two 9×9 grids that give the results of the sums and products of pairs of elements of F_9. For each of the elements, give their additive and multiplicative inverses.

(c) Are there other roots of $x^2 - 2$ in F_9? If so, list them; if not, explain why not.

∗ **3.** Find all solutions of the following system of equations for which v, w, x, y, and z are in the field $\mathbb{R}$.

$$\begin{cases} v+3w-x+y+2z &= 0\\ 2v+5w-3x-2y+4z &= 0\\ v+5w+2x+2y+z &= 0\end{cases}$$

∗ **4.** Find all solutions of the following system of equations for which w, x, y, and z are in the field $\mathbb{R}$.

$$\begin{cases} w \phantom{{}+x} + y + z &= 2\\ \phantom{w+{}} x + y + 2z &= 1\\ -w + x - y + 2z &= 1\\ w - x \phantom{{}+y} - z &= 1\end{cases}$$

∗ **5.** Find all solutions of the following system of equations for which w, x, y, and z are in the field F_3.

$$\begin{cases} w \phantom{{}+x} + y + z &= 2\\ \phantom{w+{}} x + y + 2z &= 1\\ 2w + x + 2y + 2z &= 1\\ w + 2x \phantom{{}+y} + 2z &= 1\end{cases}$$

January 23

(NOTE: There will be a quiz on the complex numbers, which is the material addressed in problems 6, 7, 8, and 9 below, the last 10 minutes of class on January 23.)

6. Let $z = 4-5i$.
Find: (a) $\mathrm{Re}(z)$ (b) $\mathrm{Im}(z)$ (c) $|z|$ (d) $\bar{z}$.

7. Compute:

(a) $(3+2i)(2-i)+i(-2+i)$ (b) $(2-3i)^2(4+2i)$

(c) $(2-i)^2+(1+3i)^2$ (d) $\left(\overline{(2-i)}\right)^2+\left(\overline{(1+3i)}\right)^2$ *see (c)*

(e) $\dfrac{1}{3+4i}$ (f) $\dfrac{4-2i}{1+i}$

(g) $\dfrac{2+3i}{(2-i)^2}+\dfrac{i}{1+i}$ (h) $\left|\dfrac{1+3i}{(2-i)}\right|$.

8. Find all of the complex numbers that deserve to be called $\sqrt{5-2i}$. How many are there?

9. Find all (3) roots of the equation $z^3-3z^2+7z-5=0$.