Mock Test 1

63. (0708) Let $\mathcal{V}$ be a finite dimensional vector space over the field $\mathbb{F}$. Show that the vectors $v_1, v_2, \cdots, v_k$ are a basis for $\mathcal{V}$ if and only if, for any non-zero linear functional f in the dual of $\mathcal{V}$, there is v_j with $1 \le j \le k$ for which $f(v_j) \ne 0$.

64. (0708) Suppose A and B are complex $n \times n$ matrices for which $AB = 0$. Prove that $\text{rank}(A + B) \le \text{rank}(A) + \text{rank}(B) \le n$.

65. (1108) Let S be a linear transformation on the finite dimensional vector space $\mathcal{V}$ over the field $\mathbb{F}$ that satisfies $S^m = S$ for some positive integer $m > 1$.

(a) Letting $\mathcal{N}(S)$ and $\mathcal{R}(S)$ be the nullspace and range of S, prove that $\mathcal{N}(S) \cap \mathcal{R}(S) = (0)$

(b) Prove that $\mathcal{N}(S) = \mathcal{N}(S^k)$ for every positive integer k.

66. An $n \times n$ matrix N over a field $\mathbb{F}$ is called *a nilpotent matrix of order* k if $N^k = 0$, but $N^{k-1} \ne 0$. For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

are nilpotent matrices of order 3 and 2 respectively. Let N be an $n \times n$ matrix that is nilpotent of order 4. Find the inverse of $I + 2N + N^2 - 3N^3$.

67. We have seen that for $\mathcal{V}$ a finite dimensional vector space over the field $\mathbb{F}$, T a linear transformation on $\mathcal{V}$, and v a vector in $\mathcal{V}$, then $J = \{p \in \mathbb{F}[x] : p(T)v = 0\}$ is an ideal in the ring of polynomial over $\mathbb{F}$.

(a) Suppose $\mathcal{V}$ is a 3-dimensional vector space over the field $\mathbb{F}$, v is a vector in $\mathcal{V}$, and T is a linear transformation on $\mathcal{V}$. Prove that there is a polynomial, p, of degree 3 for which $p(T)v = 0$.

(b) Let $D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and let $v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. Show that there is no (monic) polynomial of degree 2 in the ideal $K = \{q \in \mathbb{R}[x] : q(D)v = 0\}$.

(c) Find a monic polynomial of degree 3 in the ideal K in part (b) and explain why this means that the polynomial you just found is the monic generator of K.

(d) Let $C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$. Choose two linearly independent vectors u and w in $\mathbb{R}^3$.

Let $K_u = \{q \in \mathbb{R}[x] : q(C)u = 0\}$ and $K_w = \{q \in \mathbb{R}[x] : q(C)w = 0\}$ be the ideals in $\mathbb{R}[x]$ associated with C and u and w. For each of the ideals K_u and K_w, find a monic polynomial of degree 3 in the ideal.