January 25

$*$ **13.** Let $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix}$ Find a row reduced echelon form matrix R that is row equivalent to A and an invertible 3×3 matrix P so that $R = PA$.

$*$ **14.** An $n \times n$ matrix A is said to be an *upper triangular* if every entry below the main diagonal is 0. Prove that such a matrix A is invertible if and only if every entry on the main diagonal is non-zero.

$*$ **15.** Let A be an $n \times n$ matrix. Prove:

(a) If A is invertible and B is an $n \times n$ matrix for which $AB = 0$, then $B = 0$.

(b) If A is not invertible, there is B a non-zero $n \times n$ matrix for which $AB = 0$.

$**$ **16.** Suppose A is a square matrix with entries in a field F partitioned as

$$A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

where X and Z are square matrices and 0 is a zero matrix.

(a) Find necessary and sufficient conditions on X, Y, and Z so that A is invertible and then find formulas for P, Q, R, and S so that A^{-1} is the block matrix

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

(b) Use your formula to find A^{-1}

when $X = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 3 & -2 & 5 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -3 & 1 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

$**$ **17.** Suppose A is an $n \times n$ real matrix, u and v are real column and row vectors, respectively, and a is a real number. The matrix

$$X = \begin{pmatrix} A & u \\ v & a \end{pmatrix}$$

is an $(n+1) \times (n+1)$ matrix called a *bordered matrix.*

(a) Show that if A is invertible and $a - vA^{-1}u \neq 0$ then X is invertible and

$$X^{-1} = \begin{pmatrix} B & p \\ q & b \end{pmatrix}$$

where $B = A^{-1} + bA^{-1}uvA^{-1}$, $p = -bA^{-1}u$, $q = -bvA^{-1}$, and $b = (a - vA^{-1}u)^{-1}$

(b) Use the formula above to find the inverse of

$$\begin{pmatrix} 1 & -2 & 2 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$