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Math 53000 **Test I** **February 27, 2014**

There are 5 questions, 5 pages, and 100 points on this test.
No calculators, No books, No notes, Ask for scrap paper if you need it, $\cdots$ Test ends at 4:20p.

(20 points) 1. Let f be a complex-valued function defined on an open set G in the plane and let a be a point in G.

(a) Define: The function f is *complex differentiable at* a.

(b) Let $G = \mathbb{C}$, let $a = 1$, and let $f(z) = i\text{Re}(z^2)$.

The function f IS IS NOT complex differentiable at $a = 1$. *(circle one!!)*

Using the Definition (but *NOT* consequences of the definition like the Cauchy-Riemann Equations, etc.), prove that your assertion about the differentiability at $a = 1$ of the function f is correct.

(20 points) 2. (a) Find a linear fractional map that takes the open disk with center $1 + i$ and radius $\sqrt{2}$, that is, $G = \{z : |z - (1 + i)| < \sqrt{2}\}$, onto the half-plane $H = \{w : \text{Re}(w) < 1\}$. Explain how you know the function you created accomplishes this goal.

(b) Find the inverse of the linear fractional map you created above.

(20 points) 3. Let g be defined by the power series $g(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(z + 1)^n$

(a) Find the largest open set E on which the power series for h converges absolutely.

(b) Prove that $g(z)$ is a real number when z is in the set $E \cap \mathbb{R}$. *(Give reasons that justify your calculations!)*

(20 points) 4. (a) Write the functions $\cos(z)$ and $\sin(z)$ in terms of the exponential function.

(b) Describe the cosine function on the imaginary axis, that is, describe the curve $\cos(it)$ as t varies from $-\infty$ to ∞.

(c) Describe the sine function on the imaginary axis, that is, describe the curve $\sin(it)$ as t varies from $-\infty$ to ∞.

(20 points) 5. Let p be the polynomial $p(z) = z^2 + z - 6$.

(a) Find the power series for p centered at $z = 2$, that is, find the coefficients a_n so that $p(z) = \sum_{n=0}^{\infty} a_n(z - 2)^n$.

(b) Observe that $p(2) = 0$. Verify a fact about p that implies there is a branch q of the inverse of p that is holomorphic in a neighborhood of 0 so that $q(0) = 2$.

(c) Find the first four coefficients of the power series for this branch of the inverse of p, that is, find coefficients b_0, b_1, b_2, and b_3 so that $q(w) = \sum_{k=0}^{\infty} b_k w^k$ is holomorphic in a neighborhood of 0, satisfies $q(0) = 2$, and $q(p(z)) = z = 2 + (z - 2) + 0(z - 2)^2 + 0(z - 2)^3 + \cdots$.